A STUDY OF A LEARNING SET HIERARCHY ENCOUNTERED IN LEARNING THE CONCEPT OF THE LIMIT OF A FUNCTION

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TABLE OF CONTENTS

LIST OF TABLES

Table		Page

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Man's knowledge of the world in which he lives is constantly expanding and new discoveries are being made in many fields almost every day. Psychologists and educators have long been concerned with the task of learning and transmitting the ever-increasing amounts of knowledge and, as a result of the search for more effective methods of learning and teaching, many theories and ideas have been developed. Individual differences in learning ability have received considerable attention in recent years. The use of programmed learning materials and computer assisted instruction, which usually employs programmed material, has received much consideration as an attempt to allow for individual differences.

The ever-increasing knowledge in the field of mathematics is causing revision of the mathematics curriculum at all levels. Material that once was taught at the college level is now being taught at the high school level, or even lower. Students are now expected to learn more mathematics and at a younger age than in the past, and thus new and more effective methods of learning and teaching mathematics are needed and are being sought. When new theories and methods of teaching mathematics are proposed, efforts should be made to test the validity and applicability of these theories and methods. In the present study, it is the writer's intent to provide some information about the validity and applicability of a theory which involves both programmed learning materials and a supposedly effective procedure for learning a mathematical task.

Nature of Problem

Several studies, concerning the learning of intellectual skills having a presumed ordered relationship to each other, have been conducted during the last few years. Many of these studies have been completed by Robert M. Gagne (7) and his co-workers and have usually involved preschool or primary school children and the use of programmed instructional techniques. The present study is an attempt to determine if certain phenomena and relationships reported in some of these previous studies might be observed when the subjects are of college age and the learning program concerns the concept of the limit of a function. The concept of the limit of a function is a basic concept in calculus and is frequently defined as follows:

> The real number L is the limit of the function f, as x approaches the real number a, if for every real number ε (epsilon) greater than zero, there exists a real number δ (delta) greater than zero such that $|f(x) - L| < \varepsilon$ for every real number x satisfying the inequality $0 < |x-a| < \delta$.

Gagne (7) theorizes that certain learning tasks can be arranged in a hierarchical order of learning sets. Each learning set represents an intellectual skill and generates a substantial amount of positive transfer to a higher order intellectual skill. Gagne (4) emphasizes that the learning set must be a statement of what the individual is capable of doing and not a statement of what the individual knows.

The lowest level of the hierarchy is represented by relatively simple or general learning sets and the highest level is represented by relatively complex learning sets. The hierarchical order of the learning sets seems to imply that for a given learning set to be achieved it is desirable or essential that all relevant subordinate learning sets must have been achieved. In this sense, Gagne and Paradise (7) state:

> In order for learning to occur at any point in the hierarchy, according to this theory, each of the learning sets subordinate to a given task must be highly recallable, and integrated by a thinking process into the solution of the problem posed by the task.

Gagne (3, 6, 7) has reported a high degree of transfer of learning from one level of a learning hierarchy to the next. Merril (10), in opposition, reported that the results of his study seem to indicate that mastery of one level of a hierarchy may not be necessary for mastery of the next higher level. Mastery, in Merril's study, however, involved amount of time and number of errors made on a posttest of the material.

Gagne (3, 6, 7, 8) reports that his scheme of organizing material to be learned into a hierarchy of learning sets has been effective in producing learning. Kingsley and Hall (9) report a significant training effect in the training of conservation using a learning hierarchy. They claimed success in the training of conservation, although many other researchers had apparently failed.

Each student that attempts to master a learning hierarchy will presumably possess some intellectual skills. Some of these skills may not be relevant to the hierarchy, others may be some of the learning sets at the lowest level of the hierarchy, while still others may be some of the learning sets at various higher levels of the hierarchy. Each student presumably also possesses some degree of intelligence or a general learning rate ability. In summary of Gagne's theory to account for individual differences in rate of completion and achievement in learning programs Gagne and Paradise (7) suggest:

> . . . differences in rate of completion of a learning program are primarily dependent upon the number and kind of learning sets (i.e., the knowledge) the learner brings to the situation, secondarily upon his standing in respect to certain relevant basic abilities and not in any direct sense upon a general <u>learning rate</u> ability.

Gagne's verification of this statement (7) consisted of computing correlation coefficients among the factors mentioned.

If the conclusions inferred by Gagne are correct, then some duplication of the results should be possible. In an attempt to do this, the present study deals with some of the factors proposed by Gagne as they are related to a learning hierarchy concerning the concept of the limit of a function.

Statement of Problem

Is there any correspondence between factors proposed by Gagne concerning hierarchies of learning sets, and measures of performance on a learning program involving the concept of the limit of a function?

The factors of irrelevant basic abilities, relevant basic abilities, achievement of previously acquired learning sets, rate of learning, intelligence, and achievement of learning sets, were defined by the writer in reference to a learning hierarchy concerning the concept of the limit of a function. To determine if these factors demonstrate relationships similar to those observed by Gagne, this study considers three main questions: (1) Are there statistically significant associations among the factors? (2) Are the observed associations more pronounced at the upper levels of the hierarchy? (3) Is there a greater degree of association between performance measures and relevant basic abilities than between performance measures and irrelevant basic abilities?

Need and Pertinence of the Study

A mathematical task was chosen by the writer for several reasons. Mathematical tasks were used in many of the studies conducted by Gagne. Mathematics is largely a set of intellectual skills, and thus a learning hierarchy can be more readily derived. Mathematics, particularly college mathematics, is the area of emphasis of the writer, and thus interest and ability to program the material might be enhanced. The concept of the limit of a function is of particular importance to college mathematics.

The concept of the limit of a function is usually encountered for the first time by the student in the early stages of his first calculus course. This concept is basic to most of the mathematics that follows. In fact, with this concept, the student may prove theorems and solve problems that cannot be completed by mere algebraic manipulations. It is the writer's opinion, formed by personal experience, teaching the concept, and through discussion with students and colleagues, that many students do not fully understand this concept on their first encounter with it, and thus are handicapped in their further study of mathematics. Due to increased enrollments in college and pressures to cover more material, less class time can be devoted to discussion of the concept and fewer questions can be answered, thus creating a need for an effective self-instructional technique which is readily available to the students. The intent of the writer is to produce self-instructional materials that could be made available for those students who experience difficulty in learning the concept of the limit of a function. If the learning program, organized according to Gagne's plan of a hierarchy of learning sets, is found to be effective in producing the learning of the concept of the limit of a function it will be a worthy contribution to the teaching of college mathematics.

Operational Definitions

The following terms and symbols are used in a special way in this study.

A. Learning set: A learning set represents an intellectual skill and is a statement of what an individual is capable of doing. In particular for this study a learning set will be one of the components in the learning hierarchy which is defined below.

B. Learning hierarchy: A learning hierarchy is an ordered set of intellectual skills or learning sets. The learning hierarchy for this study orders the learning sets in such a way that learning set I represents the final task or highest level of the hierarchy, and learning sets VIIA, VIIB, and VIIC represent level VII, the lowest level of the hierarchy. The structure of this learning hierarchy is discussed further in Chapter III. Figure 1 is a pictoral representation of this learning hierarchy with each learning set identified.

C. Subordinate learning set: Since a learning hierarchy is an ordered collection of learning sets, it might be possible to think of all sets below a given learning set in the hierarchy as subordinate learning sets. For the purposes of this study a learning set will be

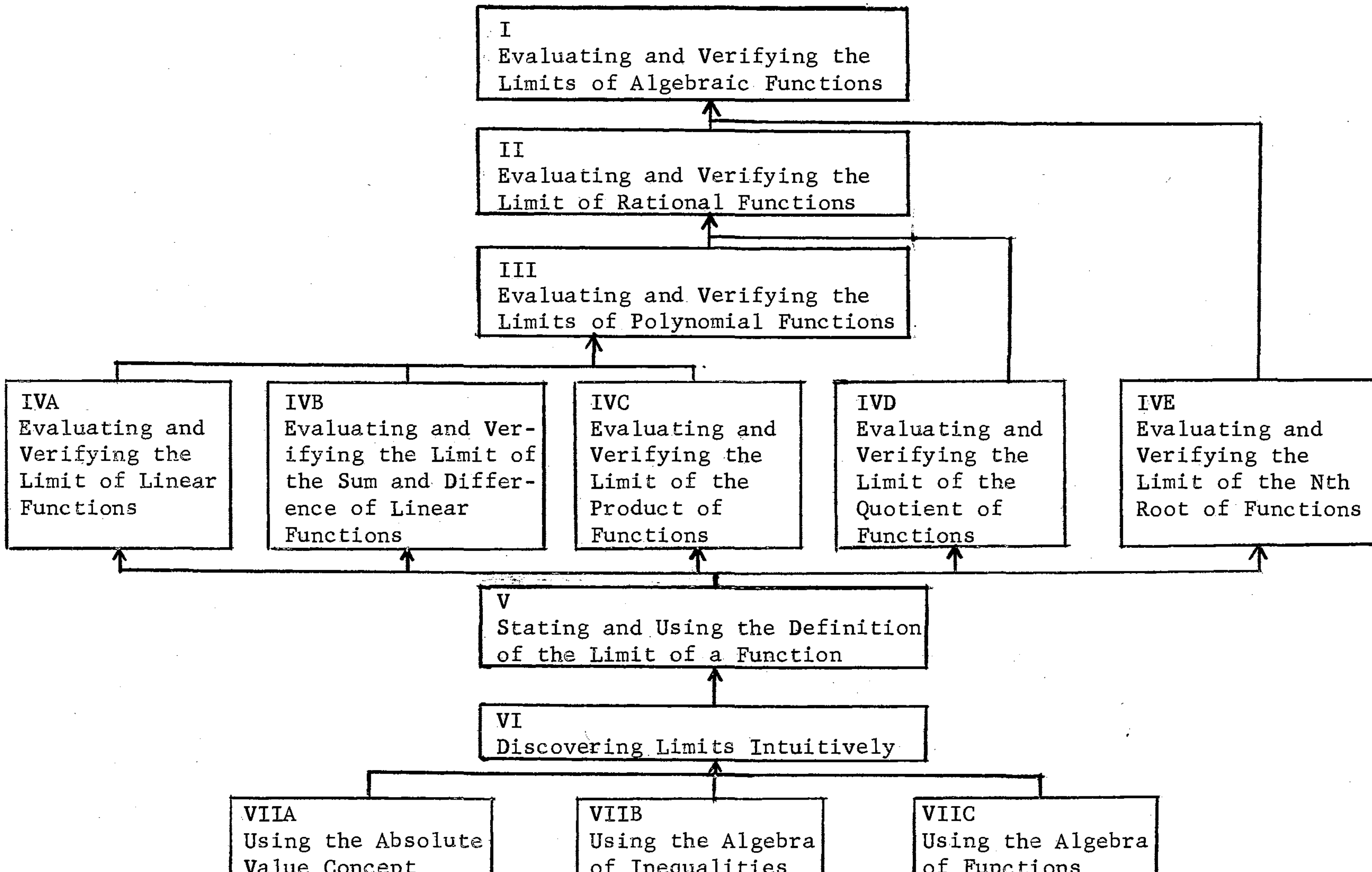

Figure 1. A Learning Hierarchy for the Concept of the Limit of a Function

called subordinate to a given learning set if the two learning sets are not separated by an intervening learning set in the hierarchy. In reference to Figure 1 this means that the line connecting a learning set and its subordinate learning set must not pass through any other learning sets.

D. Relevant basic abilities (RBA): Intellectual skills that are prerequisites of a learning task. When a learning task is organized to form a learning hierarchy relevant basic abilities are those learning sets that appear at the lowest level of the hierarchy. Relevant basic abilities for the present study therefore are VIIA, using the absolute value concept; VIIB, using the algebra of inequalities; and VIIC, using the algebra of functions.

E. Irrelevant basic abilities (IBA): Intellectual skills that are deemed to be of approximately the same difficulty level as relevant basic abilities but are not identified in the process of deriving a learning hierarchy. Therefore, the measure of an irrelevant basic ability for the present study is the Social Science test of the American College Testing Program (ACT).

F. Achievement: This word is used in the present study to denote degree of mastery of learning sets. The measure of achievement for each learning set will be a test consisting of five questions, and thus achievement scores for each learning set may vary from zero to five. The notation used to denote achievement of a particular learning set will be the letter A followed by a roman numeral which represents the number of the learning set. For example: AII denotes achievement of learning set II. ASLS denotes achievement of subordinate learning sets. Prior Achievement (PA): Achievement of learning sets a student demonstrates before beginning a learning program, refers to the pretest of learning sets. The achievement score in this case is obtained by adding the number of correct responses on each of the ten learning sets and thus the score may vary from zero to fifty.

G. Intelligence (INT): The measure of intelligence used for the present study is the composite score on the ACT test. The ACT test is designed to be a measure of a student's academic potential (1) and "Scores on the intelligence tests correlate quite highly with ACT test scores . . ." according to Munday (11).

H. Rate of learning: Rate of learning for each learning set is the number of minutes required to complete the programmed material relating to each particular learning set. Since a faster rate of learning results in a smaller number (fewer minutes), all correlations involving rate of learning will be recorded with the sign reversed. This will allow the writer to discuss increases or decreases of correlational values without having to deal with large numbers of negative correlational values, and the resulting confusion about interpretation. This procedure was also used by Gagne (7). The notation in use to denote rate of learning of a particular learning set will be the letter R followed by a roman numeral which represents the number of the learning set. For example, RVI denotes rate of learning of learning set VI.

I. The following notation is used to indicate the correlation between the two variables: For any two variables, U and V, the correlation between U and V will be denoted by r_{UxV}. For example, $r_{RI \times RII}$ denotes the correlation between the rate of learning of learning set I and rate of learning of learning set II.

Design

The methods and procedures used in this study are discussed in detail in Chapter III, but a brief outline is presented here.

In order to gain some information regarding the problem posed in this study, five independent variables and two dependent variables were identified. The statistical analysis of the correspondence between these variables required that the correlations between the variables be obtained. It was decided, through consultation with Dr. Bee of the statistics department of Oklahoma State University, that product-moment coefficients of correlation should be computed in each case. Thus product moment coefficients of correlation will be found between the independent variables of (1) relevant basic abilities, (2) irrelevant basic abilities, (3) the achievement of relevant learning sets the students demonstrate before beginning the learning program, (4) intelligence, and (5) achievement of immediately subordinate learning sets, and the dependent variables of (1) rate of learning and (2) achievement of learning sets. These correlational values will be tested for significance using table EE, page 406, Peatman (12). To determine if associations between the variables are more pronounced at upper levels of the hierarchy differences between correlational values will be tested for significance using a method given by Peatman (12) page 309.

Differences between correlational values will also be tested for significance using the method given by Peatman (12) page 309 to determine if there is greater association between performance measures and relevant basic abilities than between performance measures and irrelevant basic abilities.

Assumptions and Limitations

A major assumption of this study is that theories and procedures developed by Gagne with school children are applicable with subjects that are of college age. It is also assumed that no loss of validity results when the composite ACT score is used as a measure of intelligence and when the social science ACT score is used as a measure of an irrelevant basic ability. The literature delineated in Chapter II and other references lend some support to these assumptions.

A major limitation of this study lies in the small number of subjects (n = 22). The reliability of the statistical procedures and possibility for generalization of the results would be expected to increase if the number of subjects were larger. Other possible limitations of this study are the validity of the learning program used in the study, the validity and correctness of the learning hierarchy, and the reliability and validity of the test of learning sets. Construction of these tests, learning program, and learning hierarchy are discussed in Chapter III.

It is realized that many factors or combinations of factors besides the factors considered in this study, may be involved in learning a hierarchical task. No attempt will be made in this study to examine these factors.

CHAPTER II

REVIEW OF LITERATURE

The present study is primarily concerned with the methods and results of the study by Gagne and Paradise (7); however, several related articles are reviewed. These studies are summarized in an attempt to familiarize the reader with some of the research concerning hierarchies of learning sets and to give a theoretical basis for the development of hypotheses.

Gagne and Paradise (7) conducted an experiment in an attempt to account for individual differences in rate of completion and achievement in learning programs. Using the outline of an existing program, they constructed a hierarchy of learning sets. Considering the final task of the program, the question was asked: "What would the individual have to know how to do in order to be able to achieve this (new) task when given only instructions?" The answer to this question provided one or more subordinate learning sets. For each of these sets the question was asked and the result was again one or more subordinate learning sets. This process was carried out on a program in solving linear algebraic equations, and 22 learning sets were identified.

The subjects were 118 seventh-grade students from two Maryland junior high schools. Before the program was administered, tests of basic abilities were given. During the program, the students were asked to mark their progress on their answer sheet at three-minute intervals. This provided a method of obtaining an estimate of learning rate. After completion of the program, tests to measure final performance and transfer were given.

In order to measure transfer among learning sets one learning set and all immediately subordinate learning sets were considered. If the student achieved the higher level learning set and also achieved the lower level learning set, a (++) was recorded. If the student achieved the higher level set but failed the lower level set, a (+-) was recorded. If the student failed the higher level set and achieved the lower level set, a (-+) was recorded. If the student failed to achieve both learning sets, a (--) was recorded. The relationships (++) and (--) are in accord with Gagne's theory and the relationship (+-) is directly opposed to this theory. The relationship (-+) is not opposed to the theory, but rather Gagne indicates that this relationship implies ineffectiveness of the program. Gagne and Paradise (7) used the quotient of the number of instances in accord with the theory (++,--) divided by the total testable instances (++,--,+-) as a measure of transfer. They found that the values ranged from .91 to 1.00.

Product-moment correlations ranging in value from .12 to .68, between measures of basic abilities and measures of final performance, were established. Correlations between relevant basic abilities and measures of performance were generally higher than correlations between irrelevant abilities and measures of performance.

Correlations between each basic ability (relevant and irrelevant) and pass-fail achievement of each learning set were obtained. Comparing correlations at the various hierarchical levels, the values of correlation between relevant abilities and learning set achievement increased as the student progressed upward in the hierarchy, but there was only a very slight increase in correlation between irrelevant basic abilities and achievement of the learning sets. Gagne implies that this is not a serious threat to his theory since in a moderately ineffective program many people will achieve only in proportion to their basic ability score.

Correlations between relevant basic abilities and rate of learning of learning sets, however, were much lower at the upper levels of the hierarchy, while there was only a slight decrease in the correlation between irrelevant abilities and rate of learning as the learner progressed upward in the hierarchy. This decrease in correlation between relevant basic abilities and rate of learning is in accord with Gagne's theory.

Correlations between learning rate of a learning set and achievement of relevant and irrelevant subordinate sets were obtained. In general there was a slight increase as the learner progressed upwards in the hierarchy. Gagne and Paradise (7) report, "Each of the correlations for relevant pairs is higher than the corresponding correlations for irrelevant pairs." Using t-test of significance it was found the differences mentioned above were significant at .05 for learning sets in the upper-half of the hierarchy.

Gagne, et al (6) conducted a study of the variables repetition and guidance in programmed learning. A learning program on addition of integers was used and a learning set hierarchy of 14 learning sets was derived. Procedures similar to those of Gagne and Paradise (7) were used to test for transfer of learning from lower level to higher level learning sets. It was found that the ratio of instances of positive transfer to total testable occurrences for each case varied from .97 to 1.00.

Another test for transfer was used in this study. It involved the use of three hierarchically related learning sets. The amount of positive transfer from lower to higher learning sets and to final tasks with and without successful achievement of intervening learning sets, was found. The values of the proportion successful on the higher level task when the intervening set was achieved ranged from .51 to .89 and when the intervening set was not achieved values ranged from .00 to .33. Tests of significance of this difference were significant beyond the .001 level.

Correlations between previous mathematics grades and the number of learning sets achieved was found to be very low, as were correlations between previous mathematics grades and each of three measures of final performance. However, correlations between the number of learning sets achieved and each of three measures of final performance were high. These results are consistent with the theory of learning set hierarchies proposed by Gagne and Paradise (7).

The variables of repetition and guidance produced no significant results except when a combination of high repetition and high guidance was contrasted with a combination of low repetition and low guidance. Gagne and his co-authors concluded that identification of additional learning sets or refinement of existing ones would be much more effective in bringing about increased amounts of learning than the other variables that they had considered.

Gagne (3) constructed a hierarchy of learning sets based on a program on finding formulas for the sum of "n" terms in a number series, and used as subjects a group of ninth-grade boys. In his analysis of transfer of knowledge from lower level to higher level learning sets he found that verifying instances in all cases were 100%.

Gagne (3) found no highly significant relationship between previous grades in algebra and performance on the final task, nor between general intelligence and final performance. He, therefore, suggests that identification of relevant learning sets is most important when designing learning programs.

In another study Gagne and the staff of the University of Maryland Mathematics Project (8) attempted to test the effect of a variety of examples to provide practice on each learning set and the effect of a time interval between the attainment of related learning sets. Five different forms of a mathematics learning program were developed for the purpose and administered to 116 sixth-graders.

The four hypotheses under consideration in this study were:

1. The attainment of each learning set at progressively higher levels of the hierarchy is dependent upon the previous attainment of relevant subordinate knowledges at the next lower level.

2. Recall of subordinate learning sets, and therefore learning of the final task, is enhanced by greater amounts of variety in the repetition of task examples during learning.

3. A learning program containing repeated task examples is superior to one containing no examples besides the frame providing initial attainment of the task.

4. Part of the advantage in presenting examples, as opposed to not presenting them, resides in the interpolation of time between attainment of one subordinate learning set and the beginning of learning of the next.

The test of the first hypothesis was a test of transfer accomplished by noting pass-fail relationships between relevant learning sets. Evidences of positive transfer were ratios of values between .95

and 1.00. However, there were no significant differences between means of performance involving the other hypotheses.

The investigators (8) suggested that,

> . . . one can affect the efficiency of the learning process quite readily by manipulating the content and sequence of material, but not at all readily by manipulating the repetitiveness and temporal spacing of this content . . .

Gagne (4), in his address to Division 15 of the American Psychological Association, reviewed the topic of learning hierarchies. He based his review on both his studies and the studies of others, and excerpts from this address are presented in the next three paragraphs.

Gagne characterizes a learning hierarchy as, ". . . an ordered set of intellectual skills, such that each entity generates a substantial amount of positive transfer to the learning of a not-previously-acquired higher-order capability." He defines the learning sets that make up the hierarchy as intellectual skills or cognitive strategies and states that learning sets are not entities of verbalizable knowledge. The learning set, according to Gagne, should not be defined as a statement of what the individual knows, but rather a statement of what the individual can do such as a description of capability for action. Learning sets are subordinate skills which transfer positively to higher order tasks. Gagne states, ". . . it is desired that the subordinate skill or skills facilitate learning to such an extent that it will occur when only verbal instructions, and no further trials of practice, are given."

The learning hierarchy, according to Gagne, is a description of the relationships of positive transfer among intellectual skills and thus should be utilized in instruction, although the hierarchy is not necessarily to be a presentation sequence for instruction.

Gagne admits that a learning hierarchy cannot represent everything that can be learned nor can it be the unique or most efficient path of learning for any given individual. He concedes that more evidence about learning hierarchies is needed and that some changes might be necessary, although he states, "What I am likely to be most obstinate about changing, however, is the basic idea of the feasibility of predicting optimal sequences of learning events." Recommendations are given for two methods of study to gain evidence about positive transfer of learning. One kind of study would involve only two levels of a learning hierarchy, and the other kind of study would involve a total hierarchy related to a limited topic.

The subject of human intellectual development is considered by Gagne (5) and a model for intellectual development is proposed and discussed. This model is based on a notion of cumulative learning. Cumulative learning, reports Gagne, involves processes of differentiation, recall, and transfer of learning by which the individual learns an ordered set of capabilities. A cumulative learning sequence or learning hierarchy pertaining to judging equalities and inequalities of volumes of liquids in rectangular containers is presented and discussed.

Although no experimental evidence is presented, Gagne states,

> The stage in which any individual learner finds himself with respect to the learning of any given new capability can be specified by describing (a) the relevant capabilities he now has; and (b) any of a number of hierarchies of capabilities he must acquire in order to make possible the ultimate combination of subordinate entities which will achieve the to-be-learned task.

Transfer of learning as an important characteristic of a cumulative learning model is considered by Gagne (5), and he discusses transfer and the potential for generalization associated with such a model.

Kingsley and Hall (9) used Gagne's learning set approach in the training of conservation to young children. Eighty-six kindergarten and first-grade students were given conservation pretest and then given individual training in conservation. The training for each subject was begun at the level of his first failure on the pretest. The student was not allowed to proceed to the next level until he demonstrated mastery of the preceding level. The subjects were given posttests at least 3 days after the last training session. Kingsley and Hall reported a significant training effect in the training of conservation.

Merrill (10) conducted a study to determine if learning and retention, of a hierarchical task, are facilitated by mastering each successive part of the material before proceeding to the next. He tested the following hypotheses.

1. If Part I is mastered, subjects are able to learn Part II faster and with fewer errors than if Part I is not mastered before proceeding to Part II, etc.

2. When the terminal test requires every subject to review previously presented materials until he is able to answer every question correctly, subjects who are required to master each successive part of the task before proceeding take less total time to master the terminal test than subjects who proceed from part to part with no requirement of mastery.

3. Subjects who are required to master each successive part of the task before proceeding retain the material better than subjects who proceed from part to part with no requirement of mastery even when the terminal test requires every subject to review previously presented materials until he is able to answer every question correctly.

A branching-type program concerning an imaginary science called the Science of Xenograde Systems was administered to each of four groups. The program was presented by means of a computer-based teaching machine system knows as SOCRATES and the material was presented in five lessons with a quiz following each lesson. A correction/review procedure was developed for each lesson and quiz. Group I was administered the correction/review procedure on both lessons and quizzes. Group II had correction/review only on lessons. Group III had correction/review only on quizzes and Group IV had no correction/review. A fifth group was used as a control group and was presented only summary statements of the lessons used for the other groups. Each of the five groups was then administered a test section which utilized the correction/review procedure. A test to determine retention was given three weeks after the first test. All of the subjects were students from undergraduate or masters-level education courses at the University of Illinois.

The results reported by Merrill indicated that none of the hypotheses could be supported. In fact, with regard to the time factor the results were in direct opposition to hypotheses two and three while there was no significant difference in the number of errors made on the test section. Merrill reported that mastery of one level of the hierarchy may not be necessary for mastery of the next level.

Development of Hypotheses

The first of three questions to be examined in the present study is concerned with the significance of correlational values between certain factors related to a learning hierarchy concerning the concept of the limit of a function. The factors as previously defined are relevant basic abilities, irrelevant basic abilities, achievement of learning sets encountered before beginning the learning program, intelligence, and achievement of subordinate learning sets. According to results and suggestions reported by Gagne (7) it is predicted that both achievement and rate of learning of each learning set is dependent primarily on factors of achievement of subordinate learning sets, achievement of learning sets encountered before beginning the learning program, and relevant basic abilities, and to a lesser extent on factors of intelligence and irrelevant basic abilities. In order to test this prediction the hypotheses suggested are stated in the null form, (H_o:1), . . ., (H_o:10).

The second question to be examined in the present study concerns correlational values at different levels of the learning hierarchy. In each case to be considered it is predicted that the results will conform to the suggestions and results reported by Gagne. To examine the second question, hypotheses eleven through twenty were designed to test for increases or decreases in the correlational values as one moves from the lower to the upper levels of the hierarchy. Gagne and Paradise (7) reported that correlation between relevant basic abilities and rate of learning decreased as the learner moved up the hierarchy. Rationale given for the above result is that relevant basic abilities mediate specific, rather than general, transfer of learning and thus rate of learning will depend decreasingly on relevant basic abilities (H_o:11). On the other hand correlations between irrelevant basic abilities and rate of learning decreased only slightly as the learner moved up the hierarchy. Rationale given for this result is that irrelevant basic abilities mediate general transfer of learning and thus the effect of irrelevant basic abilities on rate of learning remains nearly constant (H_o:12).

The theories of specific and general transfer given could be used to predict that correlation between relevant basic abilities and achievement of learning sets will decrease, while correlation between irrelevant basic abilities and achievement of learning sets will decrease only slightly or remain constant as the learner moves up the hierarchy (H_o:16, H_o:17). Gagne and Paradise (7) reported, however, that correlation between relevant basic abilities and achievement of learning sets increased as the learner moved up the hierarchy. The explanation given is, if the program is ineffective some learners will not be able to attain learning sets at the upper levels of the hierarchy. Those that do attain the upper level sets are presumably those that were able to score high on tests of relevant basic abilities. This results in an increase in correlation between relevant basic abilities and achievement of learning sets as the learner moves up the hierarchy.

Relevant learning sets, which the learner has attained in some way before beginning a learning program, mediate specific transfer of learning, and thus correlation between achievement of these learning sets with both rate of learning and achievement of each learning set should decrease as the learner moves up the hierarchy (H_o:13, H_o:18).

Intelligence, according to Gagne and Paradise (7: p.4), mediates general transfer of learning and thus correlation of intelligence scores with both rate of learning and achievement of learning sets, although apparent, will remain nearly constant as the learner moves up the hierarchy (H_o:14, H_o:19).

Achievement of immediately subordinate learning sets mediates specific transfer of learning and because learning at the upper levels of the hierarchy is less dependent on basic abilities, learning may be more dependent on achievement of subordinate learning sets. Thus, correlations of achievement and subordinate learning sets with both rate of learning and achievement of each learning set should increase as the learner moves up the hierarchy (H_o:15, H_o:20). Results of the Gagne and Paradise (7) study indicate that this does occur.

The third question concerns the association between performance measures and relevant basic abilities as compared to the association between performance measures and irrelevant basic abilities. Hypotheses (H_o:21) and (H_o:22) were formulated in an attempt to answer this question. Gagne and Paradise (7) report that in most cases, the correlation between achievement of learning sets and scores on relevant basic ability tests was systematically higher than the correlation between achievement of learning sets and scores on irrelevant basic ability tests. However, the opposite trend was observed between correlation of rate of learning and relevant and irrelevant basic abilities. That is, the differences between correlation of learning rate and relevant basic abilities and correlation of learning rate and irrelevant basic abilities became smaller as the learner moved upward in the hierarchy. It is predicted that similar results will be observed in the present study.

The hypotheses suggested by the preceding discussion are now stated in the null form.

Hypotheses

(H_o:1) For each learning set, there is no significant correlation between rate of learning of the learning set and test scores of relevant basic ability tests.

(H_o:2) For each learning set, there is no significant correlation between rate of learning of the learning set and test scores of irrelevant basic ability tests.

(H_o:3) For each learning set, there is no significant correlation between rate of learning of the learning set and achievement of relevant learning sets a student demonstrates before he begins the learning program.

(H_o:4) For each learning set, there is no significant correlation between rate of learning of the learning set and intelligence as measured by ACT scores.

(H_o:5) For each learning set, there is no significant correlation between rate of learning of the learning set and achievement of the immediately subordinate learning set (or sets).

(H_o:6) For each learning set, there is no significant correlation between achievement of the learning set and test scores of relevant basic ability tests.

(H_o:7) For each learning set, there is no significant correlation between achievement of the learning set and test scores of irrelevant basic ability tests.

(H_o:8) For each learning set, there is no significant correlation between achievement of the learning set and achievement of relevant learning sets a student demonstrates before he begins the learning program.

(H_o:9) For each learning set, there is no significant correlation between achievement of the learning set and intelligence as measured by ACT scores.

(H_o:10) For each learning set, there is no significant correlation between achievement of the learning set and achievement of the immediately subordinate learning set (or sets).

(H_o:11) When considering the correlational values between relevant basic abilities and rate of learning of learning sets, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:12) When considering the correlational values between irrelevant basic abilities and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:13) When considering the correlational values between achievement of relevant learning sets a student demonstrates before he begins the learning program and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:14) When considering correlational values between intelligence and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:15) When considering the correlation values between achievement of immediately subordinate learning sets and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:16) When considering the correlational values between relevant basic abilities and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:17) When considering the correlational values between irrelevant basic abilities and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:18) When considering the correlational values between achievement of relevant learning sets the student demonstrates before beginning the learning program and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:19) When considering the correlational values between intelligence and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:20) When considering the correlational values between achievement of each learning set and its immediately subordinate learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

(H_o:21) When considering values of correlation between rate of learning and relevant basic abilities and values of correlation between rate of learning and irrelevant basic abilities at corresponding levels of the hierarchy, there is no significant difference between the values.

(H_o:22) When considering values of correlation between achievement of a learning set and relevant basic abilities and values of correlation between achievement of a learning set and irrelevant basic abilities at corresponding levels of the hierarchy, there is no significant difference between these values.

CHAPTER III

METHODS AND PROCEDURES

This chapter contains information concerning general procedures, subjects, special instruments, and specific statistical procedures used to collect and analyze the data.

Subjects

The subjects were 22 students enrolled in Mathematics 1813, Analytic Geometry, taught by the writer at Oklahoma State University during the summer session in 1968. This was the only section of Analytic Geometry taught during that session. Most of the subjects were freshman students, majoring in engineering. Since the concept of the limit of a function is not usually included in an Analytic Geometry course, no attempt was made to determine which students, if any, were repeating the course and all 23 students enrolled in the course began the experiment. One student did not take the posttest and was omitted from the study.

The subjects were told that participation in a study using programmed material was a requirement of the course and that questions concerning the subject matter of the programmed material would be included on the final test for the course. The subjects were also informed that a pretest and a posttest would be given over the programmed material but that their scores on these tests would not be used in determining their final grade in the course. It was hoped that these procedures would help to minimize the Hawthorne effect.

The Learning Hierarchy

In order to derive a hierarchy of learning sets, Gagne (4, 7) recommends that a final task be stated. Subordinate skills or learning sets are then identified by determining what prerequisites are needed to be able to learn the final task. Each of these subordinate learning sets is then considered and their subordinate learning sets are identified, and this process is continued until the entire hierarchy has been identified. Evaluating and verifying the limit of algebraic functions was chosen as a final task for the learning hierarchy used in the present study. Subordinate learning sets were identified to form the hierarchy in Figure 1 (page 7). The analysis was terminated when the learning sets of Level VII were identified since ability to use the absolute value concept, the algebra of functions, and the algebra of inequalities is usually considered prerequisite to beginning the first course in Calculus. Thus, the learning sets VIIA, using the absolute value concept; VIIB, using the algebra of inequalities; and VIIC, using the algebra of functions, were considered as relevant basic abilities and so are subordinate to learning set VI, discovering limits intuitively. Learning set VI was found to be subordinate to learning set V, stating and using the definition of the limit of a function, which in turn, was determined to be subordinate to the five learning sets of level four. Learning sets IVD, evaluating and verifying the limit of the quotient of functions and IVE, evaluating and verifying the limit of the nth root of functions, were placed at level four even though they are not directly subordinate to learning set III, as shown by the lines in Figure 1. This was done since they are related to learning set V in nearly the same fashion as are learning sets IVA, IVB, IVC, and because they are considered by the experience of the writer to be of the same level of difficulty as learning sets IVA, IVB, and IVC. Learning sets IVA, evaluating and verifying the limit of linear functions, IVB, evaluating and verifying the limit of the sum of functions, and IVC, evaluating and verifying the limit of the product of functions, were determined to be subordinate to learning set III, evaluating and verifying the limit of polynomial functions. Learning sets III and IVD were found to be subordinate to learning set II, evaluating and verifying the limit of rational functions, and II along with IVE were determined to be subordinate to the final task, learning set I, evaluating and verifying the limit of algebraic functions. In view of reports by Gagne (4, 5) the learning hierarchy presented should not be considered as unique nor does it necessarily represent everything that takes place as a student learns about the concept of the limit of a function. However, it is hoped that the learning hierarchy presented here does represent a most probable sequence of learning for any given group of learners.

The Program

Programmed material, called a learning program, designed to guide or direct learning of the learning sets of the hierarchy used in the present study was prepared by the writer. The programmed materials were prepared under the direction of Dr. Hampton and Dr. Goff, members of the doctoral committee. The procedure followed included writing a job analysis, a task analysis, behavioral objectives, criterion tests, flow chart and outline, and the actual frames of the program. As each section of frames for the program was completed the frames were reviewed by Dr. Goff. Appropriateness and mathematical content were criticized and revisions were made in light of these criticisms.

To conduct the initial testing and evaluation of the program the writer employed a method recommended by Dr. Carlton Downing of the National Society for Programmed Instruction (2). Five students were selected on the basis of ACT scores. Two high, one medium, and two low ability students were chosen. The program writer observed these five students as each progressed through the program. He recorded information concerning the frames of the program derived both from his observations and from comments of the student who was completing the program. Using this information the program was revised to eliminate errors.

Pretest-Posttest

The writer constructed a written test to determine mastery of each of the ten learning sets in the hierarchy and the three learning sets called relevant basic abilities. The test for each learning set consisted of five items. An attempt was made to order the five items according to difficulty so that the first item was least difficult and the fifth item was the most difficult. The entire test was reviewed and criticized by Dr. Goff and revised in consideration of this criticism. The entire test was given as a pretest, while the same test, excluding the first three parts relating to the relevant basic abilities, was given as a posttest.

Procedures

In order to carry out the study of factors proposed by Gagne and their relationships to the learning hierarchy concerning the concept of the limit of a function, the following procedures were employed. First, the particular terms were defined in relation to the hierarchy. Second, the scope of the study in regard to the limit concept was determined. The limit concept is one of the first concepts presented in the beginning calculus course and thus the students would normally have completed the prerequisites of this course when beginning the study of the limit concept. It was, therefore, decided for the purposes of this study, that the learning hierarchy and corresponding learning program would be written under the assumption that the subjects had completed prerequisites for the calculus course. Thus, subjects would need to have nearly completed the analytic geometry course or to have just begun the calculus course. With the permission and advice of the Chairman of the Mathematics Department, Dr. Jewett, it was decided to choose subjects from the analytic geometry course. Since extending the limit concept beyond the limit of algebraic functions would require introduction of several new concepts, the learning hierarchy and corresponding learning program were not extended beyond the topic of limits of algebraic functions.

Third, construction of the learning hierarchy, learning program, and tests of learning sets, each discussed under a separate heading in this chapter, was carried out.

Fourth, the experiment was conducted. The experiment was begun two weeks before the end of the semester, and the pretest was administered on a Monday during the regular class period. At the beginning of the class period the next day, the subjects were given instructions in use of the programmed material and were told to write the words "Record the Time" in the margins of the programmed material at the end of ten specified frames. The subjects were then told to record the time whenever they reached these points while working through the program. Since these instructions concerning time might have led to undue hurrying by the subjects, they were instructed to work at a comfortable rate and that sufficient time was allotted so that each could complete the program. The time was then announced, the subjects were told to record the time, and to begin the program. At the end of the class period the subjects were told to record the time and the programmed material was collected. On successive days during the regular class period the subjects were given their programmed materials and told to record the time and to continue working on the program, recording the time at the specified frames. At the end of each period the time was again recorded and the materials collected. Some of the subjects completed the program during the fifth class period and all subjects had completed the program by the end of the sixth class period of work on the program. The posttest was administered during the next regular class period.

After collecting the data from the experiment, the ACT composite score and ACT social science scores were obtained from the students' college records. Thus, for each of twenty-two subjects, data had been collected on each of the following twenty-four variables: RI, RII, RIII, RIVA, RIVB, RIVC, RIVD, RIVE, RV, RVI, AI, AII, AIII, AIVA, AIVB, AIVC, AIVD, AIVE, AV, AVI, RBA, IBA, PA, and INT. The notation for these variables was explained in Chapter I, and the variables are listed in the appendix. Intercorrelations were obtained among all twenty-four variables.

The first of the three main questions of this study concerned significance of relationships between variables and led to ten hypotheses. To test these hypotheses the appropriate correlational values were tested for significance using the table EE, page 406, given by Peatman (12).

Hypothesis (H_o:1) states: For each learning set, there is no significant correlation between rate of learning of the learning set and test scores of relevant basic ability tests. To test this hypothesis the correlational values $r_{RVI \times RBA}$, $r_{RV \times RBA}$, $r_{RIVA \times RBA}$, $r_{RIVB \times RBA}$, $r_{RIVC \times RBA}$, $r_{RIVD \times RBA}$, $r_{RIVE \times RBA}$, $r_{RIII \times RBA}$, $r_{RII \times RBA}$, and $r_{RI \times RBA}$ were checked for significance.

Hypothesis (H_o:2) states: For each learning set, there is no significant correlation between rate of learning of the learning set and test scores of irrelevant basic ability tests. To test this hypothesis the correlational values $r_{RVI \times IBA}$, $r_{RV \times IBA}$, $r_{RIVA \times IBA}$, $r_{RIVB \times IBA}$, $r_{RIVC \times IBA}$, $r_{RIVD \times IBA}$, $r_{RIVE \times IBA}$, $r_{RIII \times IBA}$, $r_{RII \times IBA}$, and $r_{RI \times IBA}$ were checked for significance.

Hypothesis (H_o:3) states: For each learning set, there is no significant correlation between rate of learning of the learning set and achievement of relevant learning sets a student demonstrates before he begins the learning program. To test this hypothesis the correlational values $r_{RVI \times PA}$, $r_{RV \times PA}$, $r_{RIVA \times PA}$, $r_{RIVB \times PA}$, $r_{RIVC \times PA}$, $r_{RIVD \times PA}$, $r_{RIVE \times PA}$, $r_{RIII \times PA}$, $r_{RII \times PA}$, and $r_{RI \times PA}$ were checked for significance.

Hypothesis (H_o:4) states: For each learning set, there is no significant correlation between rate of learning of the learning set and intelligence as measured by ACT scores. To test this hypothesis the correlational values $r_{RVI \times INT}$, $r_{RV \times INT}$, $r_{RIVA \times INT}$, $r_{IVB \times INT}$, $r_{RIVC \times INT}$, $r_{RIVD \times INT}$, $r_{RIVE \times INT}$, $r_{RIII \times INT}$, $r_{RII \times INT}$, and $r_{RI \times INT}$ were checked for significance.

Hypothesis (H_o:5) states: For each learning set, there is no significance correlation between rate of learning of the learning set and achievement of the immediately subordinate learning set (or sets). To test this hypothesis the correlational values $r_{RVI \times ASLS}$, $r_{RV \times ASLS}$, $r_{RIVA \times ASLS}$, $r_{RIVB \times ASLS}$, $r_{RIVC \times ASLS}$, $r_{RIVD \times ASLS}$, $r_{RIVE \times ASLS}$, $r_{RIII \times ASLS}$, $r_{RII \times ASLS}$, and $r_{RI \times ASLS}$ were checked for significance.

Since the learning sets subordinate to learning set VI are relevant basic abilities, the correlation $r_{RVI \times ASLS}$ was taken to be the correlation $r_{RVI \times RBA}$. The correlation $r_{RV \times ASLS}$ was taken to be the correlation $r_{RV \times AVI}$ and so on as indicated in Table II, page 45 of this study. Since learning set III has three directly subordinate learning sets, the correlation $r_{RII \times ASLS}$ was taken to be the arithmetic mean of the correlations $r_{RIII \times AIVA}$, $r_{RIII \times AIVB}$, and $r_{RIII \times AIVC}$. For similar reasons the correlations $r_{RII \times ASLS}$ is taken to be the arithmetic mean of the correlations $r_{RII \times AIII}$ and $r_{RII \times RIVD}$. Likewise, the correlation $r_{RI \times ASLS}$ is taken to be the arithmetic mean of the correlations $r_{RI \times AII}$ and $r_{RI \times AIVE}$.

Hypothesis (H_o:6) states: For each learning set, there is significant correlation between achievement of the learning set and test scores of relevant basic ability tests. To test this hypothesis the correlational values $r_{AVI \times RBA}$, $r_{AV \times RBA}$, $r_{AIVA \times RBA}$, $r_{AIVB \times RBA}$, $r_{AIVC \times RBA}$, $r_{AIVD \times RBA}$, $r_{AIVE \times RBA}$, $r_{AIII \times RBA}$, $r_{AII \times RBA}$, and $r_{AI \times RBA}$ were checked for significance.

Hypothesis (H_o:7) states: For each learning set, there is no significant correlation between achievement of the learning set and test scores of irrelevant basic ability tests. To test this hypothesis the correlational values $r_{AVI \times IBA}$, $r_{AV \times IBA}$, $r_{AIVA \times IBA}$, $r_{AIVB \times IBA}$, $r_{AIVC \times IBA}$, $r_{AIVD \times IBA}$, $r_{AIVE \times IBA}$, $r_{AIII \times IBA}$, $r_{AII \times IBA}$, and $r_{AI \times IBA}$ were checked for significance.

Hypothesis (H_o:8) states: For each learning set, there is no significant correlation between achievement of the learning set and achievement of relevant learning sets a student demonstrates before he begins the learning program. To test this hypothesis, the correlational values $r_{AVI \times PA}$, $r_{AV \times PA}$, $r_{AIVA \times PA}$, $r_{AIVB \times PA}$, $r_{AIVC \times PA}$, $r_{AIVD \times PA}$, $r_{AIVE \times PA}$, $r_{AIII \times PA}$, $r_{AII \times PA}$, and $r_{AI \times PA}$ were checked for significance.

Hypothesis (H_o:9) states: For each learning set, there is no significant correlation between achievement of the learning set and intelligence as measured by ACT scores. To test this hypothesis the correlational values $r_{AVI \times INT}$, $r_{AV \times INT}$, $r_{AIVA \times INT}$, $r_{AIVB \times INT}$, $r_{AIVC \times INT}$, $r_{AIVD \times INT}$, $r_{AIVE \times INT}$, $r_{AIII \times INT}$, $r_{AII \times INT}$ and $r_{AI \times INT}$ were checked for significance.

Hypothesis (H_o:10) states: For each learning set, there is no significant correlation between achievement of the learning set and achievement of the immediately subordinate learning set (or sets). To test this hypothesis the correlational values $r_{AVI \times ASLS}$, $r_{AV \times ASLS}$, $r_{AIVA \times ASLS}$, $r_{AIVB \times ASLS}$, $r_{AIVC \times ASLS}$, $r_{AIVD \times ASLS}$, $r_{AIVE \times ASLS}$,

$r_{AIII \times ASLS}$, $r_{AII \times ASLS}$, and $r_{AI \times ASLS}$ were checked for significance. As before, special considerations had to be made when the variable ASLS was involved. The correlation $r_{AVI \times ASLS}$ was taken to be $r_{AVI \times RBA}$; also, $r_{AIII \times ASLS}$ was taken to be the arithmetic mean of $r_{AIII \times AIVA}$, $r_{AIII \times AIVB}$, and $r_{AIII \times AIVC}$. The correlation $r_{AII \times ASLS}$ was taken to be the arithmetic mean of $r_{AII \times AIII}$ and $r_{AII \times AIVD}$. The correlation $r_{AI \times ASLS}$ was taken to be the arithmetic mean of $r_{AI \times AII}$ and $r_{AI \times AIVE}$.

The second major question of the study concerned increased correlational values at upper levels of the learning hierarchy and led to ten hypotheses.

The method for testing significant differences in correlational values is given by Peatman (12: p. 309). If x, y, and z are three variables and it is desired to know if there is a significant difference between the correlation r_{xz} and r_{yz}, a t value is computed by the following formula.

(1)

$$t = \frac{\left(r_{yz} - r_{xz}\right) \sqrt{(n-3)\left(1 + r_{xy}\right)}}{\sqrt{2\left(1 - r_{xz}^2 - r_{yz}^2 - r_{xy}^2 + 2r_{xz} \cdot r_{yz} \cdot r_{xy}\right)}},$$

when n represents the number of pairs of observations.

The critical value of t for the .05 significance level is found in table B of Peatman (12). The critical value and computed value of t were compared to determine significant differences between the correlational values.

Hypothesis (H_o:11) states: When considering the correlational values between relevant basic abilities and rate of learning of learning sets, there is no significant difference between the correlational values from adjacent levels of the hierarchy. To test this hypothesis the differences $r_{RV \times RBA} - r_{RVI \times RBA}$, $r_{RIVA \times RBA} - r_{RV \times RBA}$, $r_{RIVB \times RBA} - r_{RV \times RBA}$, $R_{RIVC \times RBA} - r_{RV \times RBA}$, $r_{RIVD \times RBA} - r_{RV \times RBA}$, $r_{RIVE \times RBA} - r_{RV \times RBA}$, $r_{RIII \times RBA} - r_{RIVA \times RBA}$, $r_{RIII \times RBA} - r_{RIVB \times RBA}$, $r_{RIII \times RBA} - r_{RIVC \times RBA}$, $r_{RII \times RBA} - r_{RIII \times RBA}$, $r_{RII \times RBA} - r_{RIVD \times RBA}$, $r_{RI \times RBA} - r_{RII \times RBA}$, and $r_{RI \times RBA} - r_{RIVE \times RBA}$, were computed and tested for significance.

Hypothesis (H_o:12) states: When considering the correlational values between irrelevant basic abilities and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy. To test this hypothesis, the differences $r_{RV \times IBA} - r_{RVI \times IBA}$, $r_{RIVA \times IBA} - r_{RV \times IBA}$, $r_{RIVB \times IBA} - r_{RV \times IBA}$, $r_{RIVC \times IBA} - r_{RV \times IBA}$, $r_{RIVD \times IBA} - r_{RV \times IBA}$, $r_{RIVE \times IBA} - r_{RV \times IBA}$, $r_{RIII \times IBA} - r_{RIVA \times IBA}$, $r_{RIII \times IBA} - r_{RIVB \times IBA}$, $r_{RIII \times IBA} - r_{RIVC \times IBA}$, $r_{RII \times IBA} - r_{RIII \times IBA}$, $r_{RII \times IBA} - r_{RIVD \times IBA}$, $r_{RI \times IBA} - r_{RII \times IBA}$, and $r_{RI \times IBA} - r_{RIVE \times IBA}$, were computed and tested for significance.

Hypothesis (H_o:13) states: When considering the correlational values between achievement of relevant learning sets a student demonstrates before he begins the learning program and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy. To test this hypothesis, the differences $r_{RV \times PA} - r_{RVI \times PA}$, $r_{RIVA \times PA} - r_{RV \times PA}$, $r_{RIVB \times PA} - r_{RV \times PA}$, $r_{RIVC \times PA} - r_{RV \times PA}$, $r_{RIVD \times PA} - r_{RV \times PA}$, $r_{RIVE \times PA} - r_{RV \times PA}$, $r_{RIII \times PA} - r_{RIVA \times PA}$, $r_{RIII \times PA} - r_{RIVB \times PA}$, $r_{RIII \times PA} - r_{RIVC \times PA}$, $r_{RII \times PA} - r_{RIII \times PA}$, $r_{RII \times PA} - r_{RIVD \times PA}$, $r_{RI \times PA} - r_{RII \times PA}$, and $r_{RI \times PA} - r_{RIVE \times PA}$, were computed and tested for significance.

Hypothesis (H_o:14) states: When considering correlational values between intelligence and rate of learning, there is no significant differences between the correlational values from adjacent levels of the hierarchy. To test this hypothesis the difference $r_{RV \times INT} - r_{RVI \times INT}$, $r_{RIVA \times INT} - r_{RV \times INT}$, $r_{RIVB \times INT} - r_{RV \times INT}$, $r_{RIVC \times INT} - r_{RV \times INT}$, $r_{RIVD \times INT} - r_{RV \times INT}$, $r_{RIVE \times INT} - r_{RV \times INT}$, $r_{RIII \times INT} - r_{RIVA \times INT}$, $r_{RIII \times INT} - r_{RIVB \times INT}$, $r_{RIII \times INT} - r_{RIVC \times INT}$, $r_{RII \times INT} - r_{RIII \times INT}$, $r_{RII \times INT} - r_{RIVD \times INT}$, $r_{RI \times INT} - r_{RII \times INT}$, and $r_{RI \times INT} - r_{RIVE \times INT}$, were tested for significance.

Hypothesis (H_o:15) states: When considering the correlational values between achievement of immediately subordinate learning sets and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy. To test this hypothesis, the differences $r_{RV \times ASLS} - r_{RVI \times ASLS}$, $r_{RIVA \times ASLS} - r_{RV \times ASLS}$, $r_{RIVB \times ASLS} - r_{RV \times ASLS}$, $r_{RIVC \times ASLS} - r_{RV \times ASLS}$, $r_{RIVD \times ASLS} - r_{RV \times ASLS}$, $r_{RIVE \times ASLS} - r_{RV \times ASLS}$, $r_{RIII \times ASLS} -$

$r_{RIVA \times ASLS}$, $r_{RIII \times ASLS} - r_{RIVB \times ASLS}$, $r_{RIII \times ASLS} - r_{RIVC \times ASLS}$, $r_{RII \times ASLS} - r_{RIVD \times ASLS}$, $r_{RI \times ASLS} - r_{RII \times ASLS}$, and $r_{RI \times ASLS} - r_{RIVE \times ASLS}$, were computed and tested for significance.

Hypothesis (H_o:16) states: When considering the correlational values between relevant basic abilities and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy. To test this hypothesis, the differences $r_{AV \times RBA} - r_{AVI \times RBA}$, $r_{AIVA \times RBA} - r_{AV \times RBA}$, $r_{AIVB \times RBA} - r_{AV \times RBA}$, $r_{AIVC \times RBA} - r_{AV \times RBA}$, $r_{AIVD \times RBA} - r_{AV \times RBA}$, $r_{AIVE \times RBA} - r_{AV \times RBA}$, $r_{AIII \times RBA} - r_{AIVA \times RBA}$, $r_{AIII \times RBA} - r_{AIVB \times RBA}$, $r_{AIII \times RBA} - r_{AIVC \times RBA}$, $r_{AII \times RBA} - r_{AIII \times RBA}$, $r_{AII \times RBA} - r_{AIVD \times RBA}$, $r_{AI \times RBA} - r_{AII \times RBA}$, and $r_{AI \times RBA} - r_{AIVE \times RBA}$ were computed and tested for significance.

Hypothesis (H_o:17) states: When considering the correlational values between irrelevant basic abilities and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy. To test this hypothesis, the differences $r_{AV \times IBA} - r_{AVI \times IBA}$, $r_{AIVA \times IBA} - r_{AV \times IBA}$, $r_{AIVB \times IBA} - r_{AV \times IBA}$, $r_{AIVC \times IBA} - r_{AV \times IBA}$, $r_{AIVD \times IBA} - r_{AV \times IBA}$, $r_{AIVE \times IBA} - r_{AV \times IBA}$, $r_{AIII \times IBA} - r_{AIVA \times IBA}$, $r_{AIII \times IBA} - r_{AIVB \times IBA}$, $r_{AIII \times IBA} - r_{AIVC \times IBA}$, $r_{AII \times IBA} - r_{AIII \times IBA}$, $r_{AII \times IBA} - r_{AIVD \times IBA}$, $r_{AI \times IBA} - r_{AII \times IBA}$, and $r_{AI \times IBA} - r_{AIVE \times IBA}$, were computed and tested for significance.

Hypothesis (H_o:18) states: When considering the correlational values between achievement of relevant learning sets the student demonstrates before beginning the learning program and achievement of each learning set, there is no significant difference between the correlation values from adjacent levels of the hierarchy. To test this hypothesis, the differences $r_{AV \times PA} - r_{AVI \times PA}$, $r_{AIVA \times PA} - r_{AV \times PA}$, $r_{AIVB \times PA} - r_{AV \times PA}$, $r_{AIVC \times PA} - r_{AV \times PA}$, $r_{AIVD \times PA} - r_{AV \times PA}$, $r_{AIVE \times PA} - r_{AV \times PA}$, $r_{AIII \times PA} - r_{AIVA \times PA}$, $r_{AIII \times PA} - r_{AIVB \times PA}$, $r_{AIII \times PA} - r_{AIVC \times PA}$, $r_{AII \times PA} - r_{AIII \times PA}$, $r_{AII \times PA} - r_{AIVD \times PA}$, $r_{AI \times PA} - r_{AII \times PA}$, and $r_{AI \times PA} - r_{AIVE \times PA}$, were computed and tested for significance.

Hypothesis (H_o:19) states: When considering the correlational values between intelligence and achievement of each learning set, there are no significant differences between the correlational values from adjacent levels of the hierarchy. To test this hypothesis, the difference $r_{AV \times INT} - r_{AVI \times INT}$, $r_{AIVA \times INT} - r_{AV \times INT}$, $r_{AIVB \times INT} - r_{AV \times INT}$, $r_{AIVC \times INT} - r_{AV \times INT}$, $r_{AIVD \times INT} - r_{AV \times INT}$, $r_{AIVE \times INT} - r_{AV \times INT}$, $r_{AIII \times INT} - r_{AIVA \times INT}$, $r_{AIII \times INT} - r_{AIVB \times INT}$, $r_{AIII \times INT} - r_{AIVC \times INT}$, $r_{AII \times INT} - r_{AIII \times INT}$, $r_{AII \times INT} - r_{AIVD \times INT}$, $r_{AI \times INT} - r_{AII \times INT}$, and $r_{AI \times INT} - r_{AIVE \times INT}$, were computed and tested for significance.

Hypothesis (H_o:20) states: When considering the correlational values between achievement of each learning set and its immediately subordinate learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy. To test this hypothesis, the differences $r_{AV \times ASLS} - r_{AVI \times ASLS}$, $r_{AIVA \times ASLS} - r_{AV \times ASLS}$, $r_{AIVB \times ASLS} - r_{AV \times ASLS}$, $r_{AIVC \times ASLS} - r_{AV \times ASLS}$, $r_{AIVD \times ASLS} - r_{AV \times ASLS}$, $r_{AIVE \times ASLS} - r_{AV \times ASLS}$, $r_{AIII \times ASLS} - r_{AIVA \times ASLS}$, $r_{AIII \times ASLS} - r_{AIVB \times ASLS}$, $r_{AIII \times ASLS} - r_{AIVC \times ASLS}$, $r_{AII \times ASLS} - r_{AIII \times ASLS}$, $r_{AII \times ASLS} - r_{AIVD \times ASLS}$, $r_{AI \times ASLS} - r_{AII \times ASLS}$, and $r_{AI \times ASLS} - r_{AIVE \times ASLS}$, were computed and tested for significance.

The differences in correlational values were tested for significance to determine if there was significant increase or decrease in correlational values between the dependent variables rate of learning and achievement of the learning set and the independent variables RBA, IBA, PA, INT, and ASLS, as one moved upward in the hierarchy. In some cases a learning set had more than one subordinate learning set. In these cases the process was applied separately for each subordinate learning set.

The third major question of the study concerned correlations between performance measures and relevant basic abilities as compared to correlations between performance measures and irrelevant basic abilities and led to the hypotheses (H_o:21) and (H_o:22).

Hypothesis (H_o:21) states: When considering values of correlation between rate of learning and relevant basic abilities and values of correlation between rate of learning and irrelevant basic abilities at corresponding levels of the hierarchy, there is no significant difference between the values. To test this hypothesis, differences in correlational values $r_{RVI \times RBA} - r_{RVI \times IBA}$, $r_{RV \times RBA} - r_{RV \times IBA}$,

$r_{RIVA \times RBA} - r_{RIVA \times IBA}$, $r_{RIVB \times RBA} - r_{RIVB \times IBA}$, $r_{RIVC \times RBA} - r_{RIVC \times IBA}$, $r_{RIVD \times RBA} - r_{RIVD \times IBA}$, $r_{RIVE \times RBA} - r_{RIVE \times IBA}$, $r_{RIII \times RBA} - r_{RIII \times IBA}$, $r_{RII \times RBA} - r_{RII \times IBA}$, and $r_{RI \times RBA} - r_{RI \times IBA}$, were computed and tested for significance. Each difference between correlational values was tested by computing a t value using formula (1) and comparing with the critical value of t for the .05 significance level from Table B in Peatman (12).

Hypothesis (H_o:22) states: When considering values of correlation between achievement of a learning set and relevant basic abilities and values of correlation between achievement of a learning set and irrelevant basic abilities at corresponding levels of the hierarchy, there is no significant difference between these values. To test this hypothesis, differences in correlational values $r_{AVI \times RBA} - r_{AVI \times IBA}$, $r_{AV \times RBA} - r_{AV \times IBA}$, $r_{AIVA \times RBA} - r_{AIVA \times IBA}$, $r_{AIVB \times RBA} - r_{AIVB \times IBA}$, $r_{AIVC \times RBA} - r_{AIVC \times IBA}$, $r_{AIVD \times RBA} - r_{AIVD \times IBA}$, $r_{AIVE \times RBA} - r_{AIVE \times IBA}$, $r_{AIII \times RBA} - r_{AIII \times IBA}$, $r_{AII \times RBA} - r_{AII \times IBA}$, and $r_{AI \times RBA} - r_{AI \times IBA}$, were computed and tested for significance.

Each difference between correlational values was tested by computing a t value using formula (1) and comparing with the critical value of t for the .05 significance level from Table B in Peatman (12).

CHAPTER IV

RESULTS

The information presented in this chapter concerns the results of the statistical procedures described in Chapter III.

The problem considered in this study, stated formally in Chapter I, concerned certain factors proposed by Gagne and Paradise (7) and their relationships to certain measures of performance on a learning program involving the concept of the limit of a function. Table I contains the intercorrelations between all 24 variables considered in this study. In an attempt to gain additional information about relationships between the variables involved, three questions were considered. The first of these questions concerned significance of relationships between the variables and led to ten hypotheses. Hypothesis (H_o:1) stated: For each learning set, there is no significant correlation between rate of learning of the learning set and test scores of relevant basic ability tests. The critical value of r, based on an N of twenty-two, from table EE, page 406, Peatman (12) was r = .4227. None of the correlational values involved, listed in column one of Table II, were greater than or equal to this value, and thus hypothesis (H_o:1) was not rejected for any of the learning sets.

Hypothesis (H_o:2) stated: For each learning set, there is no significant correlation between rate of learning of the learning set and test scores of irrelevant basic ability tests. The correlational

TABLE I

PRODUCT-MOMENT COEFFICIENTS OF CORRELATION AMONG 24 VARIABLES
ASSOCIATED WITH THE LEARNING HIERARCHY

	RVI	RV	RIVA	RIVB	RIVC	RIVD	RIVE	RIII	RII	RI	AVI	AV	AIVA	AIVB	AIVC	AIVD	AIVE	AIII	AII	AI	RBA	IBA	PA	INT
RVI		.642	.504	.736	.712	.300	-.013	.455	.360	.552	.052	-.185	.000	-.414	-.360	-.162	-.122	-.178	-.280	-.378	.033	.081	-.130	.114
RV			.695	.705	.784	.427	.145	.666	.492	.617	-.192	-.107	.000	-.320	-.226	.006	-.075	-.040	-.585	-.104	.064	.016	.125	-.036
RIVA				.788	.715	.415	.157	.246	.504	.500	-.217	-.067	.000	-.182	-.154	-.185	-.150	-.226	-.470	.183	.135	-.089	.134	-.013
RIVB					.739	.512	.030	.350	.614	.595	-.102	-.058	.000	-.254	-.193	-.081	-.089	-.091	-.467	-.086	.345	.061	-.019	.129
RIVC						.414	.265	.588	.639	.612	-.096	-.218	.000	-.208	-.199	.007	-.119	-.346	-.398	-.071	.091	.031	.064	-.022
RIVD							.500	.514	.570	.652	.040	.326	.000	.288	.290	-.217	.179	.062	-.264	.252	.307	.094	.524	.259
RIVE								.352	.442	.567	-.075	.029	.000	.589	.545	-.030	.241	-.086	.135	.409	-.077	-.075	.594	.029
RIII									.405	.731	-.031	.028	.000	-.030	-.046	.081	-.081	.052	-.197	-.138	-.030	.159	.230	.043
RII										.530	-.140	.020	.000	.082	.229	.222	.069	-.198	-.069	.107	.263	.135	.337	.259
RI											-.063	.041	.000	.095	.065	.112	-.062	-.057	-.250	.011	.148	.073	.371	.100
AVI												.131	.000	-.024	-.184	.066	.262	-.100	.307	.038	.308	.586	.009	.471
AV													.000	.240	.275	-.110	.054	-.110	.253	.631	.493	-.014	.416	.185
AIVA														.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
AIVB															.857	-.228	.143	.029	.106	.462	-.035	-.410	.310	-.299
AIVC																-.095	.217	-.007	.086	.382	.021	-.303	.429	-.123
AIVD																	.327	.030	.069	-.105	.336	.339	.163	.183
AIVE																		.232	-.152	.191	.221	.139	.387	.064
AIII																			-.125	-.173	-.294	-.080	.065	.020
AII																				.180	.067	.230	.116	.416
AI																					.305	-.072	.404	.094
RBA																						.300	.253	.341
IBA																							.042	.846
PA																								.196
INT																								

TABLE II

PRODUCT-MOMENT COEFFICIENTS OF CORRELATION BETWEEN RATE OF LEARNING AND FIVE MEASURES OF PERFORMANCE

	RBA	IBA	PA	INT	ASLS
RVI	.033	.081	-.130	.114	$r_{RVI \times RBA} = .033$
RV	.064	.016	.125	-.036	$r_{RV \times AVI} = -.192$
RIVA	.135	-.089	.134	-.013	$r_{RIVA \times AV} = -.067$
RIVB	.345	.061	-.019	.129	$r_{RIVB \times AV} = -.058$
RIVC	.091	.031	.064	-.022	$r_{RIVC \times AV} = -.218$
RIVD	.307	.094	*.524	.259	$r_{RIVD \times AV} = .326$
RIVE	-.077	-.075	*.594	.029	$r_{RIVE \times AV} = .029$
RIII	-.030	.159	.230	.043	$(r_{RIII \times AIVA} + r_{RIII \times AIVB} + r_{RIII \times AIVC})/3 = -.025$
RII	.263	.135	.337	.259	$(r_{RII \times AIII} + r_{RII \times AIVD})/2 = .012$
RI	.148	.073	.371	.100	$(r_{RI \times AII} + r_{RI \times AIVE})/2 = -.156$

*Significant at .05 level, critical value is r = .4227 values involved in testing hypothesis (H_o:2) were listed in column two of Table II. Since none of these correlational values were greater than or equal to the critical value, r = .4227, hypothesis (H_o:2) was not rejected for any of the learning sets.

Hypothesis (H_o:3) stated: For each learning set, there is no significant correlation between rate of learning of the learning set and achievement of relevant learning sets a student demonstrates before he begins the learning program. The correlational values involved in testing hypothesis (H_o:3) were listed in column three of Table II. Since only the correlational values $r_{RIVD \times PA} = .524$ and $r_{RIVE \times PA} = .594$ were greater than the critical value r = .4227, hypothesis (H_o:3) was rejected for learning sets IVD and IVE, and hypothesis (H_o:3) was not rejected for any of the other learning sets.

Hypothesis (H_o:4) stated: For each learning set, there is no significant correlation between rate of learning of the learning set and intelligence as measured by ACT scores. The correlational values involved in testing hypothesis (H_o:4) were listed in column four of Table II. Since none of these correlational values were greater than or equal to the critical value, r = .4227, hypothesis (H_o:4) was not rejected for any of the learning sets.

Hypothesis (H_o:5) stated: For each learning set, there is no significant correlation between rate of learning of the learning set and achievement of the immediately subordinate learning set (or sets). The correlational values involved in testing hypothesis (H_o:5) were listed in column five of Table II. Since none of these correlational values were greater than or equal to the critical value, r = .4227, hypothesis (H_o:5) was not rejected for any of the learning sets.

Hypothesis (H_o:6) stated: For each learning set, there is no significant correlation between achievement of the learning set and test scores of relevant basic ability tests. The correlational values involved in testing hypothesis (H_o:6) are listed in column one of Table III. Since only the correlational value $r_{AIV \times RBA} = .493$ was greater than or equal to the critical value, $r = .4227$, hypothesis (H_o:6) was rejected for learning set V and was not rejected for the other learning sets.

Hypothesis (H_o:7) stated: For each learning set, there is no significant correlation between achievement of the learning set and test scores of irrelevant basic ability tests. The correlational values involved in testing hypothesis (H_o:7) were listed in column two of Table III. Since only the correlational value $r_{AVI \times IBA} = .586$ was greater than or equal to the critical value, $r = .4227$, hypothesis (H_o:7) was rejected for learning set VI and was not rejected for the other learning sets.

Hypothesis (H_o:8) stated: For each learning set, there is no significant correlation between achievement of the learning set and achievement of relevant learning sets a student demonstrates before he begins the learning program. The correlational values involved in testing hypothesis (H_o:8) were listed in column three of Table III. Since only the correlational value $r_{AIVC \times PA} = .429$ was greater than or equal to the critical value, $r = .4227$, hypothesis (H_o:8) was rejected for learning set IVC and was not rejected for the other learning sets.

TABLE III

PRODUCT-MOMENT COEFFICIENTS OF CORRELATION BETWEEN ACHIEVEMENT OF LEARNING SETS AND FIVE MEASURES OF PERFORMANCE

	RBA	IBA	PA	INT	ASLS
AVI	.308	*.586	.009	*.471	$r_{AVI \times RBA} = .308$
AV	*.493	-.014	.416	.185	$r_{AV \times AVI} = .131$
AIVA	.000	.000	.000	.000	$r_{AIVA \times AV} = .000$
AIVB	-.035	-.410	.310	-.299	$r_{AIVB \times AV} = .240$
AIVC	.021	-.303	*.429	-.123	$r_{AIVC \times AV} = .275$
AIVD	.336	.339	.163	.183	$r_{AIVD \times AV} = -.110$
AIVE	.221	.139	.387	.064	$r_{AIVE \times AV} = .054$
AIII	-.294	-.080	.065	.020	$(r_{AIII \times AIVA} + r_{AIII \times AIVB} + r_{AIII \times AIVC})/3 = .007$
AII	.067	.230	.116	.416	$(r_{AII \times AIII} + r_{AII \times AIVD})/2 = -.028$
AI	.305	-.072	.404	.094	$(r_{AI \times AII} + r_{AI \times AIVE})/2 = .186$

*Significant at .05 level, critical value is r = .4227

Hypothesis (H_o:9) stated: For each learning set, there is no significant correlation between achievement of the learning set and intelligence as measured by ACT scores. The correlational values involved in testing hypothesis (H_o:9) were listed in column four of Table III. Since only the correlational value $r_{AVI \times INT} = .471$ was greater than or equal to the critical value, $r = .4227$, hypothesis (H_o:9) was rejected for learning set VI and was not rejected for the other learning sets.

Hypothesis (H_o:10) stated: For each learning set, there is no significant correlation between achievement of the learning set and achievement of the immediately subordinate learning set (or sets). The correlational values involved in testing hypothesis (H_o:10) were listed in column five of Table III. None of these correlational values were greater than or equal to the critical value, $r = .4227$, and thus hypothesis (H_o:10) was not rejected for any of the learning sets.

The experiment conducted by the writer was based on a study by Gagne (7) and thus the writer predicted results similar to those reported by Gagne. It was expected that correlational values between rate of learning of the learning sets and the factors of relevant basic abilities (RBA), prior achievement (PA), and achievement of subordinate learning sets (ASLS) would attain significance, particularly at upper levels of the hierarchy. The results recorded in Table II and test of hypotheses (H_o:1), (H_o:3), and H_o:5), did not support this prediction. In fact, six of the ten correlational values between rate of learning of a learning set and achievement of the subordinate learning set were negative.

It was also expected that correlational values between achievement of each learning set and factors of relevant basic abilities (RBA), prior achievement (PA), and achievement of subordinate learning sets (ASLS) would attain significance. However, the results recorded in Table III and tests of hypotheses (H_o:6), (H_o:8), and (H_o:10), did not support this expectation. It was also predicted that correlational values between rate of learning of each learning set and the factors irrelevant basic abilities (IBA) and intelligence (INT) would not attain significance except perhaps at the lower levels of the hierarchy. The correlational values between achievement of each learning set and the factors of irrelevant basic abilities (IBA) and intelligence (INT) also were expected to be non-significant except perhaps at the lower levels of the hierarchy. The results recorded in Tables II and III and tests of hypotheses (H_o:2), (H_o:4), (H_o:7) and (H_o:9), support these expectations. In general, however, the predicted answer, to the first major question, that both achievement and rate of learning of each learning set is dependent primarily on factors of achievement of subordinate learning sets, prior achievement, and relevant basic abilities, and to a lesser extent on factors of intelligence and irrelevant basic abilities, was not supported since most of the correlational values involved were non-significant.

The second major question that was investigated in this study involved comparing the correlational values at the different levels of the hierarchy and led to ten hypotheses. Each hypothesis was tested by finding differences between correlational values, as indicated in Chapter III, and computing a t value from each difference. Hypothesis (H_o:11) stated: When considering the correlational values between relevant basic abilities and rate of learning of learning sets, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences in correlational values and the corresponding t value were listed in Table IV. Since none of the t values were greater than the critical value, t = 2.093, hypothesis (H_o:11) was not rejected.

It had been predicted, based on a study by Gagne and Paradise (7) that correlational values between relevant basic abilities and rate of learning would decrease as the learner moved up the hierarchy. The results recorded in column one of Table II and the fact that hypothesis (H_o:11) was not rejected both indicate that the above prediction was not supported.

Hypothesis (H_o:12) stated: When considering the correlational values between irrelevant basic abilities and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences between the correlational values and the corresponding t values involved in testing hypothesis (H_o:12) were listed in Table V. Since none of the t values were greater than the critical value, t = 2.093, hypothesis (H_o:12) was not rejected.

It was predicted that correlational values between irrelevant basic abilities and rate of learning would remain nearly constant as the learner moved up the hierarchy. These correlation values are recorded in column two of Table II. Since hypotheses (H_o:12) was not rejected this prediction was supported.

TABLE IV

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING RATE OF LEARNING OF 10 LEARNING SETS AND RELEVANT BASIC ABILITIES, AND THE CORRESPONDING t VALUES

	Difference	t
$r_{RV \times RBA} - r_{RVI \times RBA} = .064 - .033 =$	.031	.160
$r_{RIVA \times RBA} - r_{RV \times RBA} = .135 - .064 =$	.071	.400
$r_{RIVB \times RBA} - r_{RV \times RBA} = .345 - .064 =$	.281	1.764
$r_{RIVC \times RBA} - r_{RV \times RBA} = .091 - .064 =$	.027	.157
$r_{RIVD \times RBA} - r_{RV \times RBA} = .307 - .064 =$	.243	.542
$r_{RIVE \times RBA} - r_{RV \times RBA} = -.077 - .064 =$	-.141	-.473
$r_{RIII \times RBA} - r_{RIVA \times RBA} = -.030 - .135 =$	-.165	-.592
$r_{RIII \times RBA} - r_{RIVB \times RBA} = -.030 - .345 =$	-.376	-1.554
$r_{RIII \times RBA} - r_{RIVC \times RBA} = -.030 - .091 =$	-.121	-.587
$r_{RII \times RBA} - r_{RIII \times RBA} = .263 - (-.030) =$	.293	1.229
$r_{RII \times RBA} - r_{RIVD \times RBA} = .263 - .307 =$	-.044	-.219
$r_{RI \times RBA} - r_{RII \times RBA} = .148 - .263 =$	-.115	-.536
$r_{RI \times RBA} - r_{RIVE \times RBA} = .148 - (-.077) =$	.225	1.088

Critical value of t at .05 level is 2.093

TABLE V

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING RATE OF LEARNING OF 10 LEARNING SETS AND IRRELEVANT BASIC ABILITIES, AND THE CORRESPONDING t VALUES

		Difference	t
$r_{RV \times IBA} - r_{RVI \times IBA}$	= .016 - .081 =	-.065	-.336
$r_{RIVA \times IBA} - r_{RV \times IBA}$	= -.089 - .016 =	-.105	.592
$r_{RIVB \times IBA} - r_{RV \times IBA}$	= .061 - .016 =	.045	.256
$r_{RIVC \times IBA} - r_{RV \times IBA}$	= .031 - .016 =	.015	.099
$r_{RIVD \times IBA} - r_{RV \times IBA}$	= .094 - .016 =	.078	.319
$r_{RIVE \times IBA} - r_{RV \times IBA}$	= -.075 - .016 =	-.091	-.304
$r_{RIII \times IBA} - r_{RIVA \times IBA}$	= .159 - (-.089) =	.248	.900
$r_{RIII \times IBA} - r_{RIVB \times IBA}$	= .159 - .061 =	.098	.379
$r_{RIII \times IBA} - r_{RIVC \times IBA}$	= .159 - .031 =	.128	.624
$r_{RII \times IBA} - r_{RIII \times IBA}$	= .135 - .159 =	-.024	-.097
$r_{RII \times IBA} - r_{RIVD \times IBA}$	= .135 - .094 =	.041	.195
$r_{RI \times IBA} - r_{RII \times IBA}$	= .073 - .135 =	-.062	-.281
$r_{RI \times IBA} - r_{RIVE \times IBA}$	= .073 - (-.075) =	.148	.702

Critical value of t at .05 level is 2.093

Hypothesis (H_o:13) stated: When considering the correlational values between achievement of relevant learning sets a student demonstrates before he begins the learning program and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences between correlational values and the corresponding t values involved to test hypothesis (H_o:13) were listed in Table VI. Since none of the t values were greater than the critical value, t = 2.093, hypothesis (H_o:13) was not rejected.

It was predicted that correlational values between prior achievement and rate of learning would decrease as the learner moved up the hierarchy. The results recorded in column three of Table II and the fact that hypothesis (H_o:13) was not rejected indicate that this prediction was not supported.

Hypothesis (H_o:14) stated: When considering correlational values between intelligence and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences between correlational values and the corresponding t values involved in testing hypothesis (H_o:14) were listed in Table VII. Since none of the t values were greater than the critical value, t = 2.093, hypothesis (H_o:14) was not rejected.

It was predicted that correlational values between intelligence scores and rate of learning recorded in column four of Table II, would remain nearly constant as the learner moved up the hierarchy. Since hypothesis (H_o:14) was not rejected, this prediction was supported.

TABLE VI

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING RATE OF LEARNING OF 10 LEARNING SETS AND PRIOR ACHIEVEMENT, AND THE CORRESPONDING t VALUES

			Difference	t
$r_{RV \times PA}$	$- r_{RVI \times PA}$	= .125 - (-.130) =	.255	1.377
$r_{RIVA \times PA}$	$- r_{RV \times PA}$	= .134 - .125 =	.009	.051
$r_{RIVB \times PA}$	$- r_{RV \times PA}$	= -.019 - .125 =	-.144	-.833
$r_{RIVC \times PA}$	$- r_{RV \times PA}$	= .064 - .125 =	-.061	-.408
$r_{RIVD \times PA}$	$- r_{RV \times PA}$	= .524 - .125 =	.399	1.924
$r_{RIVE \times PA}$	$- r_{RV \times PA}$	= .594 - .125 =	.469	1.947
$r_{RIII \times PA}$	$- r_{RIVA \times PA}$	= .230 - .134 =	.096	.351
$r_{RIII \times PA}$	$- r_{RIVB \times PA}$	= .230 - (-.019) =	.249	.984
$r_{RIII \times PA}$	$- r_{RIVC \times PA}$	= .230 - .064 =	.166	.822
$r_{RII \times PA}$	$- r_{RIII \times PA}$	= .337 - .230 =	.107	.457
$r_{RII \times PA}$	$- r_{RIVD \times PA}$	= .337 - .524 =	-.187	-1.033
$r_{RI \times PA}$	$- r_{RII \times PA}$	= .371 - .337 =	.034	.167
$r_{RI \times PA}$	$- r_{RIVE \times PA}$	= .371 - .594 =	-.223	-1.300

Critical value of t at .05 level is 2.093

TABLE VII

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING RATE OF LEARNING OF 10 LEARNING SETS AND INTELLIGENCE, AND THE CORRESPONDING t VALUES

		Difference	t
$r_{RV \times INT} - r_{RVI \times INT}$	= -.036 - .114 =	-.150	-.786
$r_{RIVA \times INT} - r_{RV \times INT}$	= -.013 - (-.036) =	.023	.128
$r_{RIVB \times INT} - r_{RV \times INT}$	= .129 - (-.036) =	.165	.960
$r_{RIVC \times INT} - r_{RV \times INT}$	= -.022 - (-.036) =	.014	.093
$r_{RIVD \times INT} - r_{RV \times INT}$	= .259 - (-.036) =	.295	1.261
$r_{RIVE \times INT} - r_{RV \times INT}$	= .029 - (-.036) =	.065	.217
$r_{RIII \times INT} - r_{RIVA \times INT}$	= .043 - (-.013) =	.056	.199
$r_{RIII \times INT} - r_{RIVB \times INT}$	= .043 - .129 =	-.086	-.332
$r_{RIII \times INT} - r_{RIVC \times INT}$	= .043 - (-.022) =	.065	.313
$r_{RII \times INT} - r_{RIII \times INT}$	= .259 - .043 =	.216	.933
$r_{RII \times INT} - r_{RIVD \times INT}$	= .259 - .259 =	.000	.000
$r_{RI \times INT} - r_{RII \times INT}$	= .100 - .259 =	-.159	-.741
$r_{RI \times INT} - r_{RIVE \times INT}$	= .100 - .029 =	.071	.335

Critical value of t at .05 level is 2.093

Hypothesis (H_o:15) stated: When considering the correlational values between achievement of immediately subordinate learning sets and rate of learning, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences between correlational values and the corresponding t values involved in testing hypothesis (H_o:15) were listed in Table VIII. The difference $r_{RIVD \times ASLS} - r_{RV \times ASLS} = .518$ led to a t value of 2.419 which was greater than the critical value t = 2.093 and thus hypothesis (H_o:15) was rejected for this case. However, the other differences in correlational values listed in Table VIII did not lead to significant t values and thus hypothesis (H_o:15) was not rejected in the other cases.

It was predicted that correlational values between achivement of subordinate learning sets and rate of learning recorded in column five of Table II would increase as the learner moved up the hierarchy. Although this was true in one case, it was not true for the other cases and thus this prediction was in general not supported.

Hypothesis (H_o:16) stated: When considering the correlational values between relevant basic abilities and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences in correlational values and the corresponding t values involved in testing hypothesis (H_o:16) were listed in Table IX. The difference $r_{AIVB \times RBA} - r_{AV \times RBA} = -.528$ led to a t value of - 2.181 which was greater (in absolute value) than the critical value, t = 2.093, and hypothesis (H_o:16) was rejected in this case. None of the other t values in Table IX were significant and thus hypothesis (H_o:16) was not rejected in the other cases.

TABLE VIII

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING RATE OF LEARNING OF 10 LEARNING SETS AND ACHIEVEMENT OF SUBORDINATE LEARNING SETS AND THE CORRESPONDING t VALUES

	Difference	t
$r_{RV \times ASLS} - r_{RVI \times ASLS} = -.192 - .033 =$	-.162	-.869
$r_{RIVA \times ASLS} - r_{RV \times ASLS} = -.067 - (-.192) =$	.125	.714
$r_{RIVB \times ASLS} - r_{RV \times ASLS} = -.058 - (-.192) =$	.134	.780
$r_{RIVC \times ASLS} - r_{RV \times ASLS} = -.218 - (-.192) =$	-.026	-.177
$r_{RIVD \times ASLS} - r_{RV \times ASLS} = .326 - (-.192) =$	.518	2.419*
$r_{RIVE \times ASLS} - r_{RV \times ASLS} = .029 - (-.192) =$	.221	.752
$r_{RIII \times ASLS} - r_{RIVA \times ASLS} = -.025 - (-.067) =$	.042	.149
$r_{RIII \times ASLS} - r_{RIVB \times ASLS} = -.025 - (-.058) =$	.033	.126
$r_{RIII \times ASLS} - r_{RIVC \times ASLS} = -.025 - (-.218) =$	.193	.958
$r_{RII \times ASLS} - r_{RIII \times ASLS} = .012 - (-.025) =$	-.013	-.052
$r_{RII \times ASLS} - r_{RIVD \times ASLS} = .012 - .326 =$	-.314	-1.601
$r_{RI \times ASLS} - r_{RII \times ASLS} = -.156 - .012 =$	-.168	-.770
$r_{RI \times ASLS} - r_{RIVE \times ASLS} = -.156 - .029 =$	-.185	-.886

*Significant at .05 level since critical value ot t is 2.093

TABLE IX

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING ACHIEVEMENT OF 10 LEARNING SETS AND RELEVANT BASIC ABILITIES, AND THE CORRESPONDING t VALUES

			Difference	t
$r_{AV \times RBA} - r_{AVI \times RBA}$	=	.493 - .308 =	.185	.733
$r_{AIVA \times RBA} - r_{AV \times RBA}$	=	.000 - .493 =	-.493	-1.747
$r_{AIVB \times RBA} - r_{AV \times RBA}$	=	-.035 - .493 =	-.528	-2.181*
$r_{AIVC \times RBA} - r_{AV \times RBA}$	=	.021 - .493 =	-.472	-1.982
$r_{AIVD \times RBA} - r_{AV \times RBA}$	=	.336 - .493 =	-.157	-.592
$r_{AIVE \times RBA} - r_{AV \times RBA}$	=	.221 - .493 =	-.272	-1.017
$r_{AIII \times RBA} - r_{AIVA \times RBA}$	=	-.294 - .000 =	-.294	-.948
$r_{AIII \times RBA} - r_{AIVB \times RBA}$	=	-.294 - (-.305) =	-.259	-.848
$r_{AIII \times RBA} - r_{AIVC \times RBA}$	=	-.294 - .021 =	-.315	-1.013
$r_{AII \times RBA} - r_{AIII \times RBA}$	=	.067 - (-.294) =	.361	1.099
$r_{AII \times RBA} - r_{AIVD \times RBA}$	=	.067 - .336 =	-.269	-.913
$r_{AI \times RBA} - r_{AII \times RBA}$	=	.305 - .067 =	.238	.851
$r_{AI \times RBA} - r_{AIVE \times RBA}$	=	.305 - .221 =	.084	.307

*Significant at .05 level since critical value of t is 2.093

In light of the results reported by Gagne and Paradise (7) it was predicted that correlational values between relevant basic abilities and achievement of learning sets would increase as the learner moved up the hierarchy. The results recorded in column one of Table III and results of testing hypothesis (H_o:16) indicated that in general this prediction was not supported.

Hypothesis (H_o:17) stated: When considering the correlational values between irrelevant basic abilities and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences between correlational values and the corresponding t values involved in testing hypothesis (H_o:17) were listed in Table X. The difference $r_{AV \times IBA} - r_{AVI \times IBA} = -.600$ led to a t value of - 2.463 which was greater (in absolute value) than the critical value, t = 2.093, and thus hypothesis (H_o:17) was rejected for this case. None of the other t values in Table X were significant and hypothesis (H_o:17) was not rejected in the other cases.

It was predicted that correlational values between irrelevant basic abilities and achievement of learning sets recorded in column two of Table III would remain nearly constant as the learner moved up the hierarchy. Although a significant decrease occurred in one case the prediction was supported in general since hypothesis (H_o:17) was not rejected for the other cases.

Hypothesis (H_o:18) stated: When considering the correlational values between achievement of relevant learning sets the student demonstrates before beginning the learning program and achievement of each learning set, there is no significant difference between the

TABLE X

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING ACHIEVEMENT OF 10 LEARNING SETS AND IRRELEVANT BASIC ABILITIES, AND THE CORRESPONDING t VALUES

		Difference	t
$r_{AV \times IBA} - r_{AVI \times IBA}$	$= -.014 - .586 =$	-.600	-2.463*
$r_{AIVA \times IBA} - r_{AV \times IBA}$	$= .000 - (-.014) =$	.014	.043
$r_{AIVB \times IBA} - r_{AV \times IBA}$	$= -.410 - (-.014) =$	-.396	-1.543
$r_{AIVC \times IBA} - r_{AV \times IBA}$	$= -.303 - (-.014) =$	-.289	-1.101
$r_{AIVD \times IBA} - r_{AV \times IBA}$	$= .339 - (-.014) =$	.353	1.098
$r_{AIVE \times IBA} - r_{AV \times IBA}$	$= .139 - (-.014) =$	.153	.490
$r_{AIII \times IBA} - r_{AIVA \times IBA}$	$= -.080 - .000 =$	-.080	-.247
$r_{AIII \times IBA} - r_{AIVB \times IBA}$	$= -.080 - (-.410) =$	.330	1.135
$r_{AIII \times IBA} - r_{AIVC \times IBA}$	$= -.080 - (-.303) =$	.223	.721
$r_{AII \times IBA} - r_{AIII \times IBA}$	$= .230 - (-.080) =$	.310	.927
$r_{AII \times IBA} - r_{AIVD \times IBA}$	$= .230 - .339 =$	-.109	-.379
$r_{AI \times IBA} - r_{AII \times IBA}$	$= -.072 - .230 =$	-.302	-1.064
$r_{AI \times IBA} - r_{AIVE \times IBA}$	$= -.072 - .139 =$	-.211	-.734

*Significant at .05 level since critical value of t is 2.093

correlational values from adjacent levels of the hierarchy. The differences between correlational values and the corresponding t values involved in testing hypothesis (H_o:18) were listed in Table XI. Since none of the t values were greater than the critical value, t = 2.093, hypothesis (H_o:18) was not rejected.

It was predicted that correlational values between prior achievement and achievement of learning sets would decrease as the learner moved up the hierarchy. This prediction was not supported by the results recorded in column three of Table III nor by the fact that hypothesis (H_o:18) was not rejected.

Hypothesis (H_o:19) stated: When considering the correlational values between intelligence and achievement of each learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy. The differences between correlational values and the corresponding t values involved in testing hypothesis (H_o:19) were listed in Table XII. Since none of the t values were greater than the critical value, t = 2.093, hypothesis (H_o:19) was not rejected.

It was predicted that correlational values between intelligence scores and achievement of learning sets, recorded in column four of Table III, would remain nearly constant as the learner moved up the hierarchy. Since hypothesis (H_o:19) was not rejected, this prediction was supported.

Hypothesis (H_o:20) stated: When considering the correlational values between achievement of each learning set and its immediately subordinate learning set, there is no significant difference between the correlational values from adjacent levels of the hierarchy.

TABLE XI

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING ACHIEVEMENT OF 10 LEARNING SETS AND PRIOR ACHIEVEMENT, AND THE CORRESPONDING t VALUES

							Difference	t
$r_{RV \times PA}$	-	$r_{RVI \times PA}$	=	.416 -	.009	=	.407	1.482
$r_{RIVA \times PA}$	-	$r_{RV \times PA}$	=	.000 -	.416	=	-.416	-1.410
$r_{RIVB \times PA}$	-	$r_{RV \times PA}$	=	.310 -	.416	=	-.106	-.424
$r_{RIVC \times PA}$	-	$r_{RV \times PA}$	=	.429 -	.416	=	.013	.055
$r_{RIVD \times PA}$	-	$r_{RV \times PA}$	=	.163 -	.416	=	-.253	-.836
$r_{RIVE \times PA}$	-	$r_{RV \times PA}$	=	.387 -	.416	=	-.029	-.110
$r_{RIII \times PA}$	-	$r_{RIVA \times PA}$	=	.065 -	.000	=	.065	.201
$r_{RIII \times PA}$	-	$r_{RIVB \times PA}$	=	.065 -	.310	=	-.245	-.807
$r_{RIII \times PA}$	-	$r_{RIVC \times PA}$	=	.065 -	.429	=	-.364	-1.241
$r_{RII \times PA}$	-	$r_{RIII \times PA}$	=	.116 -	.065	=	.051	.150
$r_{RII \times PA}$	-	$r_{RIVD \times PA}$	=	.116 -	.163	=	-.047	-.153
$r_{RI \times PA}$	-	$r_{RII \times PA}$	=	.404 -	.116	=	.288	1.073
$r_{RI \times PA}$	-	$r_{RIVE \times PA}$	=	.404 -	.387	=	.017	.068

Critical value of t at .05 level is 2.093

TABLE XII

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING ACHIEVEMENT OF 10 LEARNING SETS AND INTELLIGENCE, AND THE CORRESPONDING t VALUES

	Difference	t
$r_{AV \times INT} - r_{AVI \times INT} = .185 - .471 =$	-.286	-1.083
$r_{AIVA \times INT} - r_{AV \times INT} = .000 - .185 =$	-.185	-.580
$r_{AIVB \times INT} - r_{AV \times INT} = -.299 - .185 =$	-.484	-1.866
$r_{AIVC \times INT} - r_{AV \times INT} = -.123 - .185 =$	-.308	-1.155
$r_{AIVD \times INT} - r_{AV \times INT} = .183 - .185 =$	-.002	-.006
$r_{AIVE \times INT} - r_{AV \times INT} = .064 - .185 =$	-.121	-.391
$r_{AIII \times INT} - r_{AIVA \times INT} = .020 - .000 =$	.020	.062
$r_{AIII \times INT} - r_{AIVB \times INT} = .020 - (-.299) =$	-.319	-1.046
$r_{AIII \times INT} - r_{AIVC \times INT} = .020 - (-.123) =$	.143	.443
$r_{AII \times INT} - r_{AIII \times INT} = .416 - .020 =$	.396	1.269
$r_{AII \times INT} - r_{AIVD \times INT} = .416 - .183 =$	.233	.831
$r_{AI \times INT} - r_{AII \times INT} = .094 - .416 =$	-.322	-1.205
$r_{AI \times INT} - r_{AIVE \times INT} = .094 - .064 =$	.030	.103

Critical value of t at .05 level is 2.093

The differences between correlational values and the corresponding t values involved in testing hypothesis (H_o:19) were listed in Table XIII. Since none of the t values were greater than the critical value, t = 2.093, hypothesis (H_o:20) was not rejected.

It was predicted that correlational values between achievement of subordinate learning sets and achievement of learning sets would increase as the learner moved up the hierarchy. The results recorded in column five of Table III and the fact that hypothesis (H_o:20) was not rejected both indicate that this prediction was not supported.

The results of testing hypotheses (H_o:11), (H_o:12),...(H_o:20) imply that associations between the variables are not more pronounced at the upper levels of the hierarchy.

The third major question examined in this study concerned the association between performance measures and relevant basic abilities as compared to the association between performance measures and irrelevant basic abilities and led to two hypotheses.

Hypothesis (H_o:21) stated: When considering values of correlation between rate of learning and relevant basic abilities and values of correlation between rate of learning and irrelevant basic abilities at corresponding levels of the hierarchy, there is no significant difference between the values. This hypothesis was tested by finding differences between correlational values and computing a t value as indicated in Chapter III. Table XIV gives this data and shows that none of the differences led to a significant t value ($t \geq 2.093$). Thus hypothesis (H_o:21) was not rejected and the prediction that correlations between rate of learning and relevant basic abilities would be greater than correlations between rate of learning and irrelevant basic abilities was not supported.

TABLE XIII

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING ACHIEVEMENT OF 10 LEARNING SETS AND ACHIEVEMENT OF SUBORDINATE LEARNING SETS, AND CORRESPONDING t VALUES

					Difference	t
$r_{AV \times ASLS}$	- $r_{AVI \times ASLS}$	= .131 -	.308	=	-.177	-.618
$r_{AIVA \times ASLS}$	- $r_{AV \times ASLS}$	= .000 -	.131	=	-.131	-.407
$r_{AIVB \times ASLS}$	- $r_{AV \times ASLS}$	= .240 -	.131	=	.109	.398
$r_{AIVC \times ASLS}$	- $r_{AV \times ASLS}$	= .275 -	.131	=	.144	.543
$r_{AIVD \times ASLS}$	- $r_{AV \times ASLS}$	= -.110 -	.131	=	-.241	-.715
$r_{AIVE \times ASLS}$	- $r_{AV \times ASLS}$	= .054 -	.131	=	-.077	-.246
$r_{AIII \times ASLS}$	- $r_{AIVA \times ASLS}$	= .007 -	.000	=	.007	.002
$r_{AIII \times ASLS}$	- $r_{AIVB \times ASLS}$	= .007 -	.240	=	-.233	-.751
$r_{AIII \times ASLS}$	- $r_{AIVC \times ASLS}$	= .007 -	.275	=	-.268	-.856
$r_{AII \times ASLS}$	- $r_{AIII \times ASLS}$	= -.028 -	.007	=	-.035	-.109
$r_{AII \times ASLS}$	- $r_{AIVD \times ASLS}$	= -.028 -	(-.110)	=	.082	.264
$r_{AI \times ASLS}$	- $r_{AII \times ASLS}$	= .186 -	(-.028)	=	.214	.743
$r_{AI \times ASLS}$	- $r_{AIVE \times ASLS}$	= .186 -	.054	=	.132	.460

Critical value of t at .05 level is 2.093

TABLE XIV

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING RATE OF LEARNING OF 10 LEARNING SETS AND RELEVANT BASIC ABILITIES AND CORRELATIONAL VALUES INVOLVING RATE OF LEARNING OF 10 LEARNING SETS AND IRRELEVANT BASIC ABILITIES AND THE CORRESPONDING t VALUES

	Difference	t
$r_{RVI \times RBA} - r_{RVI \times IBA} = .033 - .081 =$	-.048	-.177
$r_{RV \times RBA} - r_{RV \times IBA} = .064 - .016 =$	.048	.174
$r_{RIVA \times RBA} - r_{RIVA \times IBA} = .135 - (-.089) =$	.244	.916
$r_{RIVB \times RBA} - r_{RIVB \times IBA} = .345 - .061 =$	.284	1.115
$r_{RIVC \times RBA} - r_{RIVC \times IBA} = .091 - .031 =$	.060	.222
$r_{RIVD \times RBA} - r_{RIVD \times IBA} = .307 - .094 =$	.213	.825
$r_{RIVE \times RBA} - r_{RIVE \times IBA} = -.077 - (-.075) =$	-.002	-.007
$r_{RIII \times RBA} - r_{RIII \times IBA} = -.030 - .159 =$	-.189	-.708
$r_{RII \times RBA} - r_{RII \times IBA} = .263 - .135 =$	.128	.490
$r_{RI \times RBA} - r_{RI \times IBA} = .148 - .073 =$	.075	.279

Critical value of t at .05 level is 2.093

Hypothesis (H_o:22) stated: When considering values of correlation between achievement of a learning set and relevant basic abilities and values of correlation between achievement of a learning set and irrelevant basic abilities at corresponding levels of the hierarchy, there is no significant difference between these values. This hypothesis was also tested by finding differences between correlational values and computing a t value as indicated in Chapter III. Table XV gives this data and shows that only one of the differences, $r_{AV \times RBA} - r_{AV \times IBA} =$ .507, led to a significant t value, t = 2.189, which was greater than the critical t value, t = 2.093. Thus, except for this one case, hypothesis (H_o:22) was not rejected and the prediction that correlations between achievement of learning sets and relevant basic abilities would be greater than correlations between achievement of learning sets and irrelevant basic abilities was in general not supported.

The correlational values and differences between correlational values were in most cases not significant and thus in general there was not sufficient information to justify the writer to reject the hypotheses.

TABLE XV

DIFFERENCES BETWEEN CORRELATIONAL VALUES INVOLVING ACHIEVEMENT OF 10 LEARNING SETS AND RELEVANT BASIC ABILITIES AND CORRELATIONAL VALUES INVOLVING ACHIEVEMENT OF 10 LEARNING SETS AND IRRELEVANT BASIC ABILITIES, AND THE CORRESPONDING t VALUES

	Difference	t
$r_{AVI \times RBA} - r_{AVI \times IBA} = .308 - .586 =$	-.278	-1.283
$r_{AV \times RBA} - r_{AV \times IBA} = .493 - (-.014) =$	.507	2.189*
$r_{AIVA \times RBA} - r_{AIVA \times IBA} = .000 - .000 =$	.000	.000
$r_{AIVB \times RBA} - r_{AIVB \times IBA} = -.035 - (-.410) =$	.375	1.523
$r_{AIVC \times RBA} - r_{AIVC \times IBA} = .021 - (-.303) =$	.324	1.262
$r_{AIVD \times RBA} - r_{AIVD \times IBA} = .336 - .339 =$	-.003	-.012
$r_{AIVE \times RBA} - r_{AIVE \times IBA} = .221 - .139 =$	.082	.311
$r_{AIII \times RBA} - r_{AIII \times IBA} = -.294 - (-.080) =$	-.214	-.825
$r_{AII \times RBA} - r_{AII \times IBA} = .067 - .230 =$	-.163	-.617
$r_{AI \times RBA} - r_{AI \times IBA} = .305 - (-.072) =$	.377	1.483

*Significant at .05 level since critical value of t is 2.093

CHAPTER V

SUMMARY AND CONCLUSIONS

The problem which was considered in this study concerned relationships between certain factors which Gagne and Paradise (7) suggest are related to learning hierarchies and two measures of performance on a learning program concerning the concept of the limit of a function. The problem was considered in terms of three questions: (1) Are there statistically significant associations among the factors? (2) Are the observed associations more pronounced at the upper levels of the hierarchy? and (3) Is there a greater degree of association between performance measures and relevant basic abilities than between performance measures and irrelevant basic abilities? The hypotheses, listed at the end of Chapter II, were stated and tested in an attempt to answer the three questions. In order to test these hypotheses a learning hierarchy concerning the concept of the limit of a function was derived and in conjunction with this a learning program was written. Pretests and posttests of each learning set were constructed and then these tests and the program were administered to 22 subjects. Using data from the results of the pretest and posttest and from the students' college records, values were obtained for the independent variables of relevant basic abilities, irrelevant basic abilities, prior achievement, intelligence, achievement of subordinate learning sets, and the dependent variables of rate of learning and achievement of each learning set.

Correlations between these variables were then computed. It was found that only six of the one hundred correlational values attained a value large enough to be considered statistically different from zero. Thus, it was concluded that the answer to question one, was in general, no, the associations among the factors were not significant.

Differences between certain of the correlational values were found in order to determine if there were increases or decreases in the correlational values as the learner moved up the hierarchy. Since only three out of one hundred thirty of these differences between correlational values were found to be significant it was concluded that the answer to question two was no, the associations are not more pronounced at the upper levels of the hierarchy. No general pattern of increase or decrease of correlational values was apparent and some of the correlational values were negative.

Differences between certain correlational values were calculated to determine if relevant basic abilities were more highly correlated with the dependent variables than were irrelevant basic abilities. Since only one out of the twenty values computed was large enough to be considered significant it was concluded that the answer to question three was no, there is not a greater degree of association between performance measures and relevant basic abilities than between performance measures and irrelevant basic abilities. Since the answer to each of the three questions was negative it was necessary to conclude that there is no significant correspondence between factors proposed by Gagne, concerning hierarchies of learning sets, and measures of performance on a learning program involving the concept of the limit of a function.

A major assumption of this study was that methods and techniques used by Gagne with school children would be applicable when the subjects are of college age. If the present study successfully employed methods and techniques used by Gagne and Paradise (7), as was intended, then it would appear that the assumption is false. This would be a possible explanation of the findings but it is the writer's opinion that the assumption is true and that other explanations of the findings can be given.

Another possible explanation of the findings involves the hierarchy of learning sets. It might be that no such hierarchy of learning sets exists or perhaps the hierarchy used in this study was incorrect. Since mathematics is logically structured it is the writer's opinion that some hierarchy of learning sets concerning the concept of the limit of a function does exist but it is possible that the hierarchy used in this study was somehow inadequate.

Although it was not essential to this study, the writer computed the average gain of the subjects on the learning program used in this study and concluded that some learning did occur. Thus, as also reported by Merrill (10), the hierarchical arrangement of learning sets may not be crucially important. This could account for some of the insignificant correlational values and hence a third possible explanation of the findings.

A fourth possible explanation of the findings is that the limitations of the study were too great to allow for meaningful results. The sample size (N = 22) was perhaps too small to allow for a definite pattern to be established. The pretest-posttest may not have had adequate validity, particularly the test for learning set IVA, since all of the subjects scored the same on achievement of learning set IVA. This resulted in zero correlation between achievement of learning set IVA and each of the independent variables.

It is possible that some combination of the above explanations, perhaps including some unknown factors, would best explain the findings and conclusions of this study. The fact that no definite reason was given for the disagreement between expected findings and reported findings, and the fact that the sample was small and not randomly chosen does not allow the writer to generalize the results to a large population. In fact, the writer is reluctant to generalize the results to a population consisting of freshman students at Oklahoma State University who would enroll in the calculus course. Instead, the following recommendations will be made for future research, and if the results are consistently the same after each recommendation is employed some generalization might be attempted.

The first recommendation is to rerun the experiment changing nothing except the sample. It is recommended that a large sample size and preferably random selection of subjects be used. A second recommendation is to again rerun the experiment using a large sample size but also using improved tests of achievement of the learning sets, refinement of other measurement techniques and definitions of variables and possibly some revision and improvement of the program. A third recommendation is to make some attempt at verification of the learning hierarchy. In a recent article Gagne (4) recommends that some attempt be made to verify the placement of learning sets in the hierarchy. If the hierarchy needs revision the program must also be revised correspondingly. The experiment should then be rerun with all three of these recommendations being employed. The fourth recommendation is to analyze the results of this third rerun in the usual way and then do a further analysis of these results. It is recommended that partial correlations be computed between the variables involved. Partial correlation coefficients reflect the correlation between two variables while the effects of the other variables are held constant. This technique might give more information about the relationships between the variables than can be gained by considering only the simple correlation coefficients. If all four of these recommendations are carried out and the results still correspond to the results of the present study, it is the writer's opinion that the results could be generalized to other similar groups of freshman students and that achievement of each subordinate learning set may not be necessary for mastery of a learning hierarchy concerning the concept of the limit of a function.

In final summary, a learning program organized according to a hierarchy of learning sets produced some learning but correlations among dependent variables of rate of learning and achievement of learning sets and independent variables of relevant basic abilities, irrelevant basic abilities, prior achievement, intelligence, and achievement of subordinate learning sets were generally not significantly different from zero. This led to results not consistent with results reported by Gagne and Paradise (7). Rather than make generalizations, recommendations for future research were made. It was suggested that results of this research would allow for some generalization.

The present study concerned a mathematical topic which is encountered in the early stages of the college education of most scientists and engineers. It is the writer's opinion that, with the increased emphasis on technology, it is important that the concept of the limit of a function, as well as other basic mathematical concepts, be learned early and learned well so that the future scientists or engineers will be able to continue their study of more advanced mathematics.

It is recommended that more research be done on the teaching of the limit concept and it is the hope of the writer that those involved in teaching and educational research continue to search for more effective ways of teaching and learning mathematics.

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APPENDIX A

VARIABLES USED IN THE STUDY

RVI: Rate of learning for learning set VI.

RV: Rate of learning for learning set V.

RIVA: Rate of learning for learning set IVA.

RIVB: Rate of learning for learning set IVB.

RIVC: Rate of learning for learning set IVC.

RIVD: Rate of learning for learning set IVD.

RIVE: Rate of learning for learning set IVE.

RIII: Rate of learning for learning set III.

RII: Rate of learning for learning set II.

RI: Rate of learning for learning set I.

AVI: Achievement for learning set VI.

AV: Achievement for learning set V.

AIVA: Achievement for learning set IVA.

AIVB: Achievement for learning set IVB.

AIVC: Achievement for learning set IVC.

AIVD: Achievement for learning set IVD.

AIVE: Achievement for learning set IVE.

AIII: Achievement for learning set III.

AII: Achievement for learning set II.

AI: Achievement for learning set I.

RBA: Relevant basic abilities.

IBA: Irrelevant basic abilities.

PA: Prior achievement.

INT: Intelligence as measured by ACT omposite score.

ASLS: Achievement of subordinate learning sets.

APPENDIX B

LEARNING PROGRAM USED IN THE STUDY AND INSTRUCTIONS FOR ITS USE

"Read silently as I read aloud the first page of the material I have placed before you.

"'This set of materials is called a program, and it is supposed to help you learn. It is not a test.

"'The concept of the limit of a function is fundamental to the Calculus. This program is designed to help you gain familiarity with the concept of limit and proficiency in finding limits of simple functions and relations.

"'It is assumed that you are able to perform algebraic manipulations, that you are familiar with the idea of absolute value, and in general have all the prerequisites necessary to be taking the first course in Calculus.

"'The program consists of frames. Each frame consists of some information and a question or statement to which you must respond. After you have completed your response you are to check for the correct response. You should do this for each frame even though you feel certain that you have made a correct response. After completing your consideration of a frame go to the next frame and continue in this fashion throughout the material.

"'The correct response to each frame is presented in this space along with some explanation of the alternative responses. When you have completed your consideration of this material go to the next frame.'

"When reading this material I advise using a cover page in the following manner. Using a blank sheet of paper, cover the first page containing frames of the program. Pull the cover page down until the first line across the page can be seen. The material above this line is frame one of the program. Read frame one and write your response in the space provided. Pull the cover page down until the next line across the page can be seen. The information thus uncovered is the proper response to the first frame. If your response differs from the proper response, reread frame one and attempt to determine why your response was not correct and why the listed response is correct.

"When you have completed your consideration of frame one and the corresponding response pull the cover page down to the next line. Frame two can then be read and a response made. Pull the cover sheet to check the response as before, and then in a similar fashion continue on to frame three and the rest of the program."

1. To begin the study of limit complete the following:

(a) If $f(x) = 3x + 5$, then $f(2) =$ ________

(b) If $f(x) = \frac{x + 2}{x + 3}$, then $f(1) =$ ________

(c) If $f(x) = \frac{2x + 3}{x - 2}$, then $f(3) =$ ________

(d) If $f(x) = \frac{x - 2}{x - 3}$, then $f(4) =$ ________

(a) 11 (b) 3/4 (c) 9 (d) 2

---

2. Complete the following:

(a) If $f(x) = \frac{x - 2}{x + 2}$, then $f(2) =$ ________

(b) If $f(x) = \frac{x + 2}{x - 2}$, then $f(2) =$ ________

(c) Id $f(x) = \frac{x - 3}{2x - 6}$, then $f(3) =$ ________

(a) 0 (b) Undefined, since division by zero is not defined

(c) There is no unique answer. $\frac{0}{0} = k$ implies $0 = 0 \cdot k = 0$ for every k.

Any expression of the form $\frac{0}{0}$ is called indeterminate.

---

3. The major difficulties in evaluation of limits arises in situations like those is parts (b) and (c) of frame 2.

Given $f(x) = \frac{x - 2}{x - 3}$, then $f(3) = \frac{3 - 1}{3 - 3} = \frac{2}{0}$ and $f(3)$ is ____________.

Undefined

---

4. Which of the following are undefined? ________

(a) $f(-4)$, if $f(x) = \frac{x + 2}{x + 4}$

(b) $f(2)$, if $f(x) = \frac{(x + 3)(x + 4)}{(x - 2)(x - 3)}$

(c) $f(3)$, if $f(x) = \frac{x^2 + 7x + 12}{x^2 - 5x + 6}$

(d) $f(1)$, if $f(x) = \frac{x^2 - 2x + 1}{x^2 - 1}$

---

a, b, and c are undefined, d is indeterminate.

5. Given $f(x) = \frac{(x - 1)(x - 2)}{3(x - 2)(x + 1)}$. Then $f(2) = \frac{1 \cdot 0}{3 \cdot 0 \cdot 3} = \frac{0}{0}$ and $f(2)$ is ______

---

indeterminate

---

6. Which of the following are indeterminate? ________

(a) $f(3)$, if $f(x) = \frac{(x + 1)(x - 3)}{x + 3}$

(b) $f(3)$, if $f(x) = \frac{x^2 - 2x - 3}{x - 3}$

(c) $f(4)$, if $f(x) = \frac{(x - 3)(x - 2)}{(x - 2)(x - 4)}$

(d) $f(2)$, if $f(x) = \frac{x^2 - 5x + 6}{x^2 - 6x + 8}$

---

b and d, $f(3) = 0$ in (a) and $f(4)$ is undefined in (c)

---

7. If $f(x) = \frac{x^2 - 2x - 15}{x^2 - 25}$, then $f(x) = \frac{(x - 5)(x + 3)}{(x - 5)(x + 5)}$

Thus: (a) $f(3)$ = ________ (b) $f(5)$ = ________

(c) $f(-3)$ = ________ (d) $f(-5)$ = ________

(e) $f(0)$ = ________

(a) 6/8 (b) indeterminate (c) 0 (d) undefined (e) 3/5

---

8. From the preceeding frames it should be evident that it is not always possible to find the value $f(x)$ at a given value of the variable x. However, it is usually possible and generally desirable to determine how the function behaves as the variable x gets close to this given value.

Consider the table:

x	1.8	1.9	1.95	1.999
$3x + 5$	10.4	10.7	10.85	10.997

As x gets close to 2, the value of $3x + 5$ gets close to ________

---

9. Consider the table:

x	2	1.5	1.25	1.12	1.06	1.001
$2x - 2$	2	1	0.5	0.24	0.12	0.002

The value of $2x - 2$ gets close to or approaches ________ as x gets close to or approaches 1.

---

10. Consider the table:

2.3	2.2	2.1	2.0001	x
11.9	11.6	11.3	11.0003	$3x + 5$

The value of $3x + 5$ approaches 11 as x approaches ________.

2

---

11. The statement, the value of $3x + 5$ approaches 11 as x approaches 2 is written symbolically.

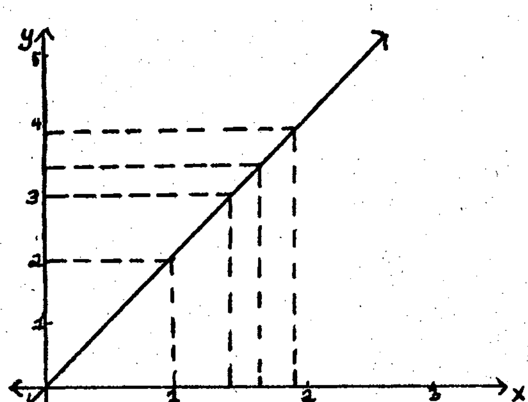
$$\lim_{x \to 2} (3x + 5) = 11.$$

Write the statement $5x + 2$ approaches 17 as x approaches 3, in symbolic form ____________.

$\lim_{x \to 3} (5x + 2) = 17$

---

12. Consider a portion of the graph of the function $f(x) = 2x$



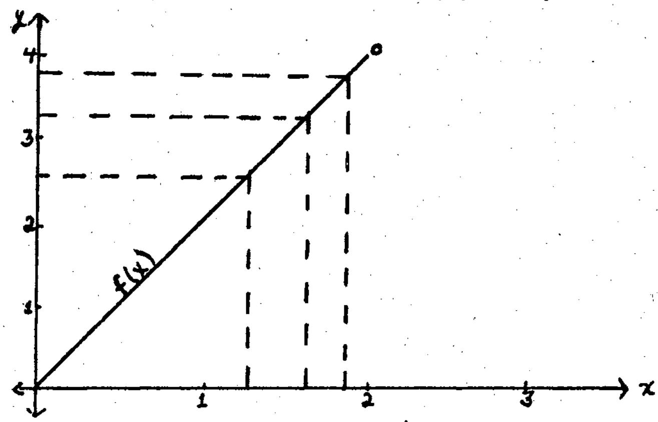
It can be seen that $f(x) = 2x$ approaches ________ as x approaches 2 and therefore that

$\lim_{x \to 2} 2x =$ ________.

---

4 4

13. Consider the graph of the function $f(x) = 2x,\ 0 \le x < 2$



It can be seen that 2x approaches 4 as x approaches ______________ and therefore that $\lim_{x \to \_\_\_\_} 2x = 4$

---

2   2   Notice that the function value approaches 4 as x approaches 2 even though the function is not defined at x = 2.

---

14. It ______________ (is - is not) necessary for a function f to be defined at a particular value of the variable x, in order to determine the limit as x approaches this value.

---

is not

---

15. (a) If $f(x) = 5x + 2,\ x \neq 4$ (defined for all values except $x = 4$), then $\lim_{x \to 4} f(x) =$ ______________; (b) If $f(x) = 5x + 2,\ x \neq 4$, and $f(4) = 10$, then $\lim_{x \to 4} f(x) =$ ______________, (c) If $f(x) = 5x + 2$ (defined for all values of x), the $\lim_{x \to 4} 5x + 2 =$ ______________; (d) If $f(x) = 5x + 2$, then $f(4) =$ ______________.

(a) 22   (b) 22   (c) 22   (d) 22

Notice that the $\lim_{x \to 4} (5x + 2) = 22$ even though in part (a) f(4) was not defined, and in part (b), f(4) was defined to be 10 rather than 22.

16. Although the formal definition of the limit of a function will be given later, the statement that the limit of $f(x)$ as x approaches a, equals b, will be written symbolically as $\lim_{x \to a} f(x) = b$. Therefore, the $\lim_{x \to c} f(x) = d$ indicates that $f(x)$ approaches the number ________ as the value of x approaches (but remains unequal to) the number ________.

---

d

---

17. Evaluate:

$\lim_{x \to 2} (3x - 1) =$ ________

$\lim_{x \to 3} \dfrac{2x - 4}{x + 2} =$ ________

---

5 2/5

Note that in both cases one does not encounter an indeterminate form.

---

18. Direct substitution of the number 2 for the value of x in the expression $\dfrac{x^2 - 4}{x - 2}$ results in:

________ (a) a real number

________ (b) 0

________ (c) division by zero which is undefined

________ (d) an indeterminate form.

(a) Incorrect. $\dfrac{4 - 4}{2 - 2} = \dfrac{0}{0}$ which is an indeterminate form.

(b) Incorrect. Although the numerator is zero the denominator is also zero. $\dfrac{0}{0}$ is an indeterminate form.

(c) Incorrect. Although the denominator is zero, the numerator is also zero and $\dfrac{0}{0}$ is an indeterminate form.

(d) Correct. $\dfrac{4 - 4}{2 - 2} = \dfrac{0}{0}$ which is an indeterminate form.

---

19. Can the $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ be found by direct substitution of 2 for x as in the examples of frame 17? ________

No, a method for evaluating limits in situations like the above is given in the next five frames.

20. Since $\frac{x^2 - 4}{x - 2}$ is determinate for $x = 2$, if the value of $\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}\right)$ is to be found, it is desired to know what happens to the expression $\frac{x^2 - 4}{x - 2}$ as x ________ 2 but is not equal to ________.

Approaches 2

---

21. $\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$ if $x \neq 2$

The above statement is ________ (true, false).

True $\frac{(x - 2)(x + 2)}{(x - 2)} = \frac{x - 2}{x - 2}(x + 2)$ and since if $x \neq 2$, then $\frac{x - 2}{x - 2} = 1$, and thus $\frac{x - 2}{x - 2} \cdot (x + 2) = 1 \cdot (x + 2) = x + 2$

---

22. Since $\frac{x^2 - 4}{x - 2} = x + 2$ if $x \neq 2$, then $\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}\right)$ ________ (is, is not) equal to $\lim_{x \to 2} (x + 2)$

---

is--since limit of f as x approaches 2 does not depend on what happens at $x = 2$.

---

23. $\lim_{x \to 2} (x + 2) =$ ________

---

4 - Here substituting 2 directly into the expression does not result in an indeterminate form.

---

24. Therefore, frames 21, 22, and 23 indicate $\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \quad 2}$ ________ $= 4$

$x + 2$

24.1 The above procedure can be outlined in the following way. To evaluate $\lim_{x \to a} f(x)$, try direct substitution of a for x. If this results in an undefined or indeterminate form, attempt to simplify the function by algebraic manipulation and factoring. Then substitute ________ for x in the new expression to evaluate the limit of the original function.

---

a, two more examples follow.

---

25. To find $\lim_{y \to 3} \frac{y - 3}{y^2 - y - 6}$, first write $\frac{y - 3}{y^2 - y - 6}$ as $\frac{y - 3}{(\quad)(\quad)}$

Note that direct substitution of 3 for y results in an indeterminate form.

$(y - 3)(y + 2)$ or $(y + 2)(y - 3)$

---

26. If $y \neq 3$ $\frac{y - 3}{(y - 3)(y + 2)}$ = ________

$\frac{1}{y + 2}$, since if $y \neq 3$ $\frac{y - 3}{y - 3} = 1$

---

27. The $\lim_{y \to 3} \frac{y - 3}{y^2 - y - 6} = \lim_{x \to 3}$ ________ = ________

$\frac{1}{y + 2}$; $\frac{1}{5}$

---

28. The $\lim_{Z \to 3} \left(\frac{\frac{1}{Z} - \frac{1}{3}}{Z - 3} \right) = \lim_{Z \to 3} \left(\frac{\frac{3 - Z}{3Z}}{Z - 3} \right) = \lim_{Z \to 3} \left(-\frac{(Z - 3)}{3Z} \right.$ ________ $\left. \right)$ Thus

$\lim_{Z \to 3} \frac{-(Z - 3)}{3Z(Z - 3)} = \lim_{Z \to 3} \left(\text{\_\_\_\_\_\_} \right) =$ ________

$\frac{1}{Z - 3}$; $-\frac{1}{3Z}$; $-\frac{1}{9}$

---

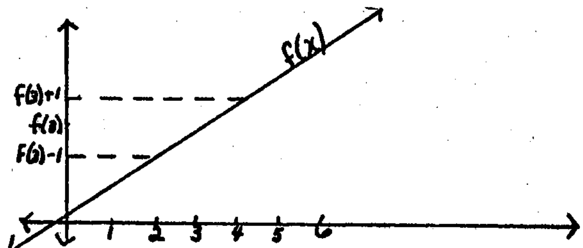
29. Use procedures of the preceeding frames to find the limit of the following:

(a) $\lim\limits_{x\to -3} \dfrac{x^2 - 9}{x + 3}$ = ________ (b) $\lim\limits_{x\to -2} \dfrac{\frac{1}{x+1} + 1}{x + 2}$ = ________

(c) $\lim\limits_{x\to 4} \dfrac{\frac{1}{x} - \frac{1}{2}}{x^2 - 4}$ = ________

(a) -6 (b) -1 (c) $-\frac{1}{16}$

30.

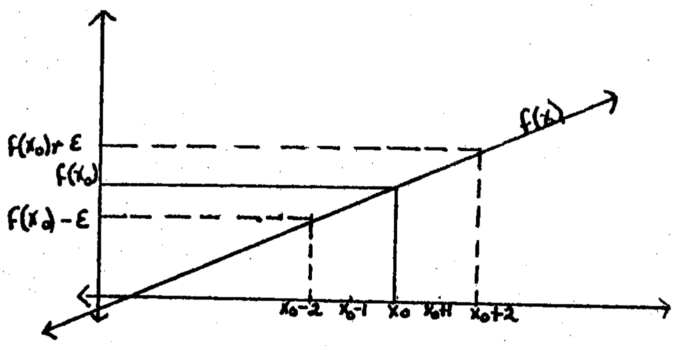


If $f(3) - 1 < f(x) < f(3) + 1$, then ________ $< x <$ ________.

---

2 4

31.



If $f(x_o) - \varepsilon < f(x) < f(x_o) + \varepsilon$, (the symbol ε is epsilon and represents a positive real number),

then x_o - ________ $< x < x_o$ + ________.

2, 2

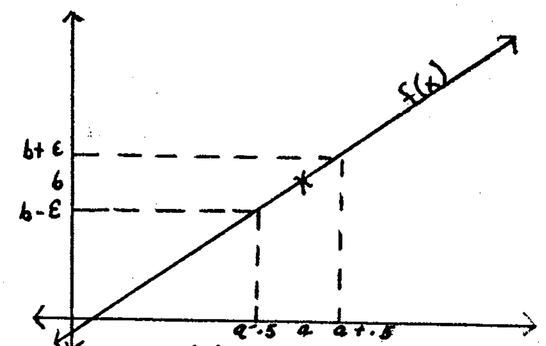
---

32. In frames 30 and 31 it was assumed that the functions denoted by $f(x)$ were defined at each value of x, however, in order to determine the limit of $f(x)$ as $x \to a$ it is not necessary for $f(x)$ to be defined at x = ________.

---

a

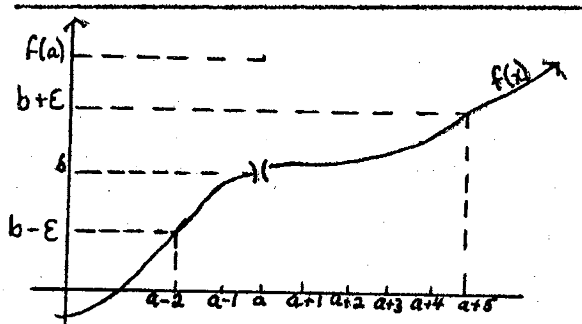
33.



Suppose that $f(x)$ is not defined at $x = a$. If $a - .5 < x < a + .5$, $x \neq a$, then $b - \mathcal{E} < f(x) <$ ________.

$b + \mathcal{E}$

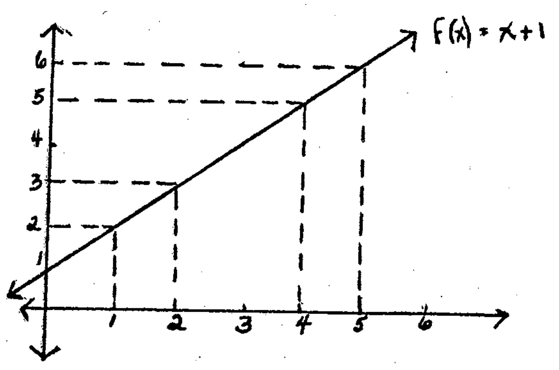
34.



If $a - 2 < x < a + 5$, $x \neq a$, then ______ $< f(x) <$ _______.

$b - \mathcal{E}$; $b + \mathcal{E}$

35.



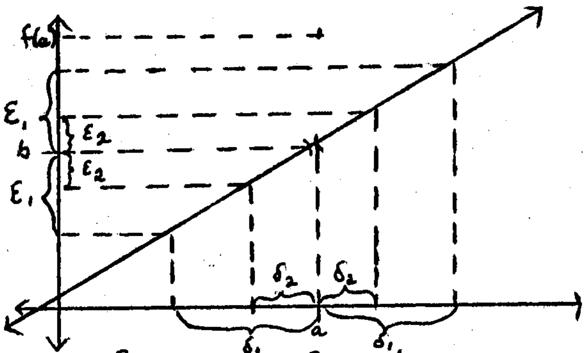
If $x \neq 3$ and $3 - 1 < x < 3 + 1$, then _________ $< f(x) <$ _________.

If $x \neq 3$ and $3 - 2 < x < 3 + 2$, then _________ $< f(x) <$ _________.

3; 5 2 ; 6

---

36. Let δ_i, ϵ_i, $i = 1,2$ be a positive real numbers. (δ is for delta)



T or F

(a) If $a - \delta_1, < x < a + \delta_1$, $x \neq a$, then $b - \epsilon_1 < f(x) < b + \epsilon_1$. ________

(b) If $a - \delta_1, < x < a + \delta_1$, $x \neq a$, then $b - \epsilon_2 < f(x) < b + \epsilon_2$. ________

(c) If $a - \delta_2, < x < a + \delta_2$, $x \neq a$, then $b - \epsilon_2 < f(x) < b + \epsilon_2$. ________

(a) T

(b) F

(c) T

---

37. Frame 36 demonstrates that $a - \delta < x < a + \delta$ (where $x \neq a$ and $\delta > o$) implies $b - \epsilon < f(x) < b + \epsilon$, and when ϵ decreases, the value of has to ________ (increase, decrease).

---

decrease

---

38. Although geometric demonstration such as the graphs in the previous frames helps one to visualize how a function behaves in the vicinity of a particular value, it is more precise mathematically to analyze this behavior algebraically.

Complete:

$|f(x) - 3| < 2$

if and only is $-2 < f(x) -3 < 2$

if and only if $-2 + 3 < f(x) < 2 + 3$

if and only if ________ $< f(x) <$ ________

---

1 ; 5 -- the if and only if means that the steps are reversible.

39. Complete:

$|f(x) + 2| < 5$

if and only if $-5 < f(x) + 2 < 5$

if and only if ______ $< f(x) <$ ______

-7 , 3

---

40. In general:

$|f(x) - c| < d$

if and only if ______ $< f(x) <$ ______

$c - d$, $c + d$ or $-d + c$, $d + c$

---

41. Suppose $f(x) = 3x$, then

$|f(x) - 9| = |3x - 9| < 1$

if and only if ______ $< 3x - 9 <$ ______

if and only if $-1/3 < x -3 <$ ______

if and only if $|(\_\_\_\_)| < 1/3$

-1, 1

$1/3$

$x - 3$

These techniques will be helpful in doing some of the following frames.

---

42. Let $f(x) = x$

If $|x - 4| < 1\frac{1}{2}$, then $|f(x) - 4| < 1$. ______ true, false

If $|x - 4| < 1\frac{1}{2}$, then $|f(x) - 4| < \frac{1}{2}$. ______ true, false

---

false

false

---

43. Let $f(x) = x$

If $|x - 4| < .9$, then $|f(x) - 4| < 1$. ______ true, false

If $|x - 4| < .9$, then $|f(x) - 4| < \frac{1}{2}$. ______ true, false

---

true

false

---

44. Let $f(x) = x$

If $|x - 4| < 2/3$, then $|f(x) - 4| < 1$. ______ true, false

If $|x - 4| < 2/3$, then $|f(x) - 4| < \frac{1}{2}$. ______ true, false

---

TRUE

false

---

45. Let $f(x) = x$

If $|x - 4| < 1/4$, then $|f(x) - 4| < 1$. ______ true, false

Id $|x - 4| < 1/4$, then $|f(x) - 4| < \frac{1}{2}$. ______ true, false

---

true

true

---

46. Considering the four (4) preceeding frames, if $\mathcal{E} \in \{1, \frac{1}{2}\}$ then there ______ (does, does not) exist a $\delta \in \{1\frac{1}{2}, .9, 2/3, 1/4\}$ such that $0 < |x - 4| < \delta$ implies $|f(x) - 4| < \mathcal{E}$.

---

does, namely the number 1/4.

---

47. There are, of course, many values for δ which would be suitable in the above frame if one were allowed to choose values other than those in the set $\{1\frac{1}{2}, .9, 2/3, 1/4\}$. In fact, every real number δ such that $0 < \delta \leq \frac{1}{2}$ would be suitable. The following 4 frames provide a method of computing suitable values δ for the special case $f(x) = 3x$.

Let $f(x) = 3x$

$|3x - 6| < 1/3$

if and only if $|3(x-2)| < 1/3$

if and only if $|3||x-2| < 1/3$

if and only if $(3)|x- 2| < 1/3$, since $|3| = 3$

if and only if $|x - 2| < (1/3)(1/3)$

if and only if $|x - 2| <$ ______

$|x - 2| < \frac{1}{9}$

---

48. Let $f(x) = 3x$

$|3x - 9| < 1/6$

If and only if $|3(x - 3)| < 1/6$

If and only if $|3||x - 3| < 1/6$

If and only if $3|x - 3| < 1/6$, since $|3| = 3$

If and only if ______________

$|x - 3| < \frac{1}{18}$

---

49. Let $f(x) = 3x$

$|3x - 3| < \frac{1}{2}$

if and only if $|3(x - 1)| < \frac{1}{2}$

if and only if __________

$|x - 1| < 1/6$

---

50. Let $f(x) = 3x$

$|3x - 6| < 1/6$

if and only if __________

$|x - 2| < \frac{1}{18}$

---

51. Let $f(x) = 3x$

(True or False)

________ (a) If $|x - 2| < 1/3$, then $|f(x) - 6| < 1/6$

________ (b) If $|x - 2| < 1/27$, then $|f(x) - 6| < 1/6$

________ (c) If $|x - 2| < 1/30$, then $|f(x) - 6| < 1/12$

________ (d) If $|x - 2| < 1/36$, then $|f(x) - 6| < 1/12$

(a) False

$|x - 2| < 1/3$ implies $|3x - 6| < 1$ or $|f(x) - 6| < 1$ so that for some x such that $|x - 2| < 1/3$ it is possible that $1/6 < |f(x) - 6| < 1$

(b) True, in fact $|x - 2| < 1/27$ implies $|f(x) - 6| < 1/9$.

(c) False, $|x - 2| < 1/30$ implies $|f(x) - 6| < 1/10$ and $1/10 \not< 1/12$.

(d) True

---

52. Let $\mathcal{E}$ (where $\mathcal{E}$ represents a positive real number) be the number such that $|f(x) - 6| < \mathcal{E}$. For the function $f(x) = 3x$, what are all values for δ such that if $\mathcal{E} = 1/6$
the statement: if $0 < |x - 2| < \delta$, then $|f(x) - 6| < \mathcal{E}$, is a true statement?

__________ $< \delta \leq$ __________

(a) $0 < \delta \leq 1/18$

---

53. Find all values of δ such that if $\mathcal{E} = 1/12$ and $f(x) = 3x$ the following statement is true. If $0 < |x - 2| < \delta$, then $|f(x) - 6| < \mathcal{E}$

__________ $< \delta \leq$ __________

$0 < \delta \leq 1/36$

---

54. In frame (52) $\mathcal{E}$ was $\frac{1}{6}$ and frame (53) $\mathcal{E}$ was $\frac{1}{12}$. The value of $\mathcal{E}$ in frame (53) is $\frac{1}{2}$ of the value of $\mathcal{E}$ in frame (52). The largest suitable value of δ for frame (52) is $\frac{1}{18}$ and the largest suitable value of δ for frame (53) is $\frac{1}{36}$. Thus, the value of δ in frame (53) is __________ of the value of δ in frame (52). Also, notice in each case that the largest suitable value of δ is $\frac{1}{3}$ of the value of $\mathcal{E}$.

$\frac{1}{2}$

---

55. The concept of limit of a function involves the idea of $f(x)$ getting close to or approaching the number b as x approaches the number a. If one continues decreasing $\mathcal{E}$ and is always able to find a δ for each $\mathcal{E}$ such that $0 < |x - a| < \delta$ implies $|f(x) - b| < \mathcal{E}$. Then $f(x)$ approaches __________ as x approaches __________. This is formalized as the definition of the limit of a function.

---

b a

---

56. Definition: The limit of the function f as x approaches, a equals b, $\lim_{x \to a} f(x) = b$, if for every real number $\mathcal{E} > 0$ there exists a real number $\delta > 0$ such that $|f(x) - b| < \mathcal{E}$ for every x (in the domain of f) satisfying the inequality $0 < |x - a| < \delta$.

No Response Requested

---

57. The definition states that a suitable number δ must be found for __________ $\epsilon > 0$.

---

every

---

58. The inequality $|f(x) - b| < \epsilon$ implies that $f(x)$ must be within ϵ units of __________.

---

b

---

59. The inequality $0 < |x - a| < \delta$ implies that x must be within δ units of __________ but __________ (can, cannot) be equal to a.

---

a, even though a may not be in the domain of f.

Cannot - The limit of a function depends on how the function behaves as x approaches a not upon what happens at x = a and thus the restriction that $0 < |x - a|$.

---

60. In frames 52 and again in 53, it was shown that for a specific value of ϵ it was possible to find a number δ, in fact an entire interval of numbers, such that if x was within δ units of a, then f(x) was within ϵ units of b. The definition states that $\lim_{x \to a} f(x) = b$ if for every $\epsilon > 0$ one can find at least __________ such number .

---

one

---

61. The statement $\lim_{x \to a} f(x) = b$ implies the limit exists and is b. To show that a limit does not exist one must show there does not exist such a real number b. Frames 61 through 71 demonstrate a method of proving that a limit does not always exist.

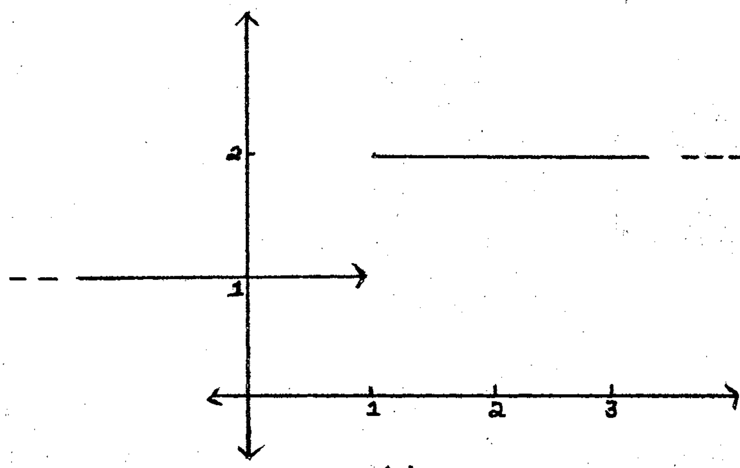
The definition of the limit of a function states that for every $\epsilon > 0$ one must produce a suitable number δ. Thus, to show that the limit of a function is <u>not</u> equal to some particular number it is necessary to exhibit at least __________ $\epsilon > 0$ such that no suitable δ can be found.

---

one

---

62. Consider the function $f(x) = \begin{cases} 1 & x < 1 \\ 2 & x \geq 1 \end{cases}$ a portion of which is shown below:



What, in your opinion is the limit of f(x) as x approaches 1?

(a) 2

(b) 1

(c) $1\frac{1}{2}$

(d) no limit exists

(a) No, the limit does not exist. Go to frame 63.

(b) No, the limit does not exist. Frame 68 demonstrates that the $\lim_{x \to 1} f(x) \neq 1$ Go first to frame 63.

(c) No the limit does not exist. It is shown in frame 69 that $\lim_{x \to 1} f(x) \neq 1\frac{1}{2}$, but go first to frame 63.

(d) Correct, frames 63-71 indicate methods of verifying this.

---

63. To show that the $\lim_{x \to 1} f(x) \neq 2$, f(x) defined in frame 62, one could use the method suggested in frame 61. That is, show that there exists at least ______ $\epsilon > 0$, for which no suitable δ can be found to satisfy the definition of the limit of a function.

---

one

---

64. Let $\epsilon = \frac{1}{2}$ and $\delta = \frac{1}{5}$, and the function f as defined in frame 62. When $x \geq 1$, $f(x) = 2$ and $0 < |x - 1| < \delta$ implies $|f(x) - 2| < \epsilon$, since $|2 - 2| =$ ___ $< \frac{1}{2}$. But $f(x) = 1$ for $x < 1$, and $0 < |x - 1| < \delta$ does not imply $|f(x) - 2| < \epsilon$ since $|1 - 2| =$ _____ $\nless \frac{1}{2}$.

---

0 1

---

65. Replace $\delta = \frac{1}{2}$ by $\delta = 1/4$ in frame 53. Again when $x \geq 1$, $0 < |x - 1| < \delta$ implies $|f(x) - 2| < \epsilon$, but for $x < 1$, $0 < |x - 1| < \delta$ _______ (does, does not) imply $|f(x) - 2| < \epsilon$, since $f(x) =$ ________ for these values of x and $|1 - 2| = 1$ and one is not less than one half.

---

does not 1

---

66. It soon becomes apparent that no value of δ is suitable since for any value of x that is less than 1, $f(x) = 1$ and theus $|f(x) - 2| = |1 - 2|$ $=$ ________ $\nless \frac{1}{2}$ for those values of x.

1

---

67. Thus, a value of ϵ has been found, namely $\epsilon = \frac{1}{2}$, such that no value of will satisfy the definition of the limit of a function.

Thus, $\lim_{x \to 1} f(x) \neq$ __________.

2

---

68. To show that the limit as x approaches one of the function defined in frame 62 is not equal to one, let $\epsilon = \frac{1}{5}$. Then no matter how small δ is chosen, for values of x such that $x \geq 1$, $f(x) = 2$ and $|f(x) - 1| = |2 - 1| =$ ______ $< \frac{1}{5}$. Thus, no suitable δ can be found and $\lim_{x \to 1} f(x)$ is __________ (equal, not equal) to 1.

---

1 Not equal

---

69. To show that the $\lim_{x \to 1} f(x)$ for this function is not equal to some number between 1 and 2 such as $1\frac{1}{2}$ let $\epsilon = 1/4$. Then $x \neq 1$, $f(x) = 1$ or $f(x) = 2$ and so $|f(x) - 1\frac{1}{2}| =$ ________.

1/2

70. Since $1/2 \nless 1/4$ the $\lim_{x \to 1} f(x)$ $\underset{(=,\ \neq)}{\underline{\qquad}}$ $1\frac{1}{2}$.

$\neq$

71. It can be shown in a similar fashion that for any real number b that a particular value of ϵ can be choosen for which there does not exist a suitable δ. This implies that $\lim_{x \to 1} f(x)$ ________ exist for this particular function.

does not

72. The definition of $\lim_{x \to a} f(x) = b$ requires that for each $\epsilon > 0$, one must exhibit a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - b| < \epsilon$. If it is possible to define δ in terms of ϵ, for example, $\delta = \epsilon$ or $\delta = \epsilon/2$, then for each value ϵ there will correspond a value δ and the definition will be satisfied. The following frames demonstrate procedures for determining values δ in terms of ϵ.

Let $f(x) = 2x$. In order to prove $\lim_{x \to 3} 2x = 6$, it is necessary to find for every $\epsilon > 0$, a $\delta > 0$ such that $0 < |x - 3| < \delta$ implies $|2x - 6| < \epsilon$.

(a) Consider $\epsilon = 1$, then $|2x - 6| < 1$ if and only if $|2(x - 3)| < 1$
if and only if $|2||x - 3| < 1$
if and only if $|x - 3| <$ ________.

(b) Now, if $0 < \delta \leq 1/2$, then $0 < |x - 3| < \delta$ implies $|2x - 6| <$ ________.

(a) 1/2 (b) 1

73. (a) Now, consider $\epsilon = 1/2$, then $|2x - 6| < 1/2$ if and only if $|2(x - 3)| < 1/2$; if and only if $|2||x - 3| < 1/2$; if and only if $|x - 3| <$ ________.

(b) Thus, if $0 < \delta \leq 1/4$, then $0 < |x - 3| < \delta$ implies $|2x - 6| <$ ________.

(a) 1/4 (b) 1/2

74. (a) Further, consider $\mathcal{E} = 1/8$, then $|2x - 6| < 1/8$ if and only if $|2(x - 3)| < 1/8$, if and only if $|x - 3| <$ ________.

(b) Thus, if $0 < \delta \le 1/16$, then $0 < |x - 3| < \delta$ implies $|2x - 6| <$ ______.

(a) 1/16 (b) 1/8

75. (a) Now in general, let $\mathcal{E}$ be any positive number, then $|2x - 6| < \mathcal{E}$ if and only if $|x - 3| <$ ________.

(b) Thus, if $0 < \delta \le \mathcal{E}/2$, then $0 < |x - 3| < \delta$ implies $|2x - 6| <$ ______.

(a) $\mathcal{E}/2$ (b) $\mathcal{E}$

76. Thus, for every $\mathcal{E} > 0$, there exists a δ namely $0 < \delta \le$ ____ which satisfies the definition of limit and thus proves that $\lim_{x \to 3} 2x =$ _____.

$\mathcal{E}/2$ 6

77. Consider $f(x) = \frac{x}{3}$, then $\lim_{x \to 9} f(x) =$ ______.

3 is correct, but this is an intuitive response and must be proven to be mathematically correct.

78. (a) Let $f(x) = \frac{x}{3}$ and consider $\mathcal{E} = 1$, then

$|f(x) - 3| = |\frac{x}{3} - 3| < 1$ if and only if $|\frac{1}{3}(x - 9)| < 1$.

if and only if $|1/3||x - 9| < 1$.

if and only if $|x - 9| <$ _____.

(b) Now, if $0 < \delta \le 3$, then $0 < |x - 9| < \delta$ implies $|x/3 - 3| <$ ______.

(a) 3 (b) 1

79. (a) Now, consider $\mathcal{E} = 1/2$, then $|x/3 - 3| < 1/2$ if and only if $|1/3||x - 9| < \frac{1}{2}$

if and only if $|x - 9| <$ ____

(b) Thus, if $0 < \delta \le 3/2$, then $0 < |x - 9| < \delta$ implies $|x/3 - 3| <$ _____.

(a) 3/2 (b) 1/2

80. (a) Further, let $\mathcal{E}$ be any positive number, then $|x/3 - 3| < \mathcal{E}$

if and only if $|x - 9| <$ ______.

(b) Thus, if $0 < \delta \le 3\mathcal{E}$, then $0 < |x - 9| < \delta$ implies $|x/3 - 3| <$ ______.

(a) $3\mathcal{E}$ (b) $\mathcal{E}$

81. Thus for every $\epsilon > 0$, there exists a δ, namely $0 < \delta \le 3\epsilon$, which satisfies the definition of limit of a function and thus proves that $\lim_{x \to \_\_} \frac{x}{3} =$ ______.

9 3

82. Consider $f(x) = 2x + 1$, then $\lim_{x \to 2} f(x) =$ ________.

5 is correct but this is an intuitive response and must be proven to be mathematically correct.

83. Let $f(x) = 2x + 1$.

(a) Consider ϵ any positive real number, then $|(2x + 1) - 5| < \epsilon$
if and only if $|2x - 4| < \epsilon$
if and only if $|x - 2| <$ _______.

(b) Thus, if $0 < \delta \le \epsilon/2$, then $0 < |x - 2| < \delta$ implies $|(2x + 1) - 5| <$ ____.

(a) $\epsilon/2$ (b) ϵ

84. Thus, for every $\epsilon > 0$, there exists a δ, namely $0 < \delta \le \epsilon/2$, which satisfies the definition of limit of a function and thus proves that $\lim_{x \to \_\_}(2x + 1) =$
________.

2 5

85. A more general form for linear functions could be
$\lim_{x \to a} (mx + b) =$ ________, $m \neq 0$, $b \neq 0$.

$ma + b$ is correct and this result will now be stated and proven as a theorem.

86. Theorem I: If $f(x) = mx + b$, where ma and b are real constants, then
$\lim_{x \to a} (mx + b) = ma + b$.
If $m \neq 0$, then for every $\epsilon > 0$, $|(mx + b) - (ma + b)| < \epsilon$ if and only if
$|m($________$)| < \epsilon$.

$x - a$

87. $|m(x - a)| < \epsilon$ if and only if ________ $\cdot |x - a| < \epsilon$

$|m|$

88. $|m||x - a| < \varepsilon$ if and only if $|x - a| <$ ________

$\varepsilon/|m|$

---

89. (a) Let $f(x) = mx + b$ and consider $\varepsilon > 0$, then by frames 86, 87, 88,
$|(mx + b) - (ma + b)| < \varepsilon$ δ if and only if $|x - a| <$ ________.
(b) Thus, if $0 < \delta \leq \frac{\varepsilon}{|m|}$, then $0 < |x + a| < \delta$ implies
$|(mx + b) - (ma + b)| <$ ________.

(a) $\varepsilon/|m|$ (b) ε

---

90. Since the condition in frame 89(b) satisfies the definition of limit of a function, this implies that $\lim_{x \to a} (mx + b) =$ ________.

$ma + b$

90.1 For example, if $f(x) = 3x + 2$, them $\lim_{x \to 2} f(x) =$ ________.

8

---

91. A special case of the function $f(x) = mx + b$ is the following:
(a) Let $m = 0$ then $f(x) = b$, and $|(mx + b) + (ma + b)| = |$ ________ $| = 0$.
Hence, $|(mx + b) - (ma + b)| < \varepsilon$ for every $\varepsilon > 0$.
(b) Thus, for any $\delta > 0$, $0 < |x - a| < \delta$ implies $|(mx + b) - (ma + b)| = |b - b| < \varepsilon$ which means that if $f(x) = b$, $\lim_{x \to a} f(x) =$ ________.

(a) $b - b$ (b) b

91.1 For example, if $f(x) = 7$, for all x, then $\lim_{x \to 3\frac{1}{2}} f(x) =$ ________.

7

---

92. Thus, the limit of a constant function for any value of x in the domain of the function is exactly that constant. Now consider another special case of the function $f(x) = mx + b$.
Let $b = 0$ and $m \neq 0$, then $f(x) =$ ________.

---

mx

---

93. (a) Let $\varepsilon > 0$, then $|mx - ma| < \varepsilon$ implies $|x - a| <$ ________.

(b) Thus, if $0 < \delta \leq \varepsilon/|m|$, then $0 < |x - a| < \delta$ implies $|mx - ma| <$ ______.

(a) $\varepsilon/|m|$ (b) ε

94. Thus, there exist a δ, namely $0 < \delta \leq \varepsilon/|m|$ that satisfies the definition of limit of a function and proves that $\lim\limits_{x \to a} mx =$ ________.

ma

94.1 For example, if $f(x) = 15x$, then the $\lim\limits_{x \to 2} f(x) =$ ________.

30

95. If m = 1 in the above argument, then there exists a $\delta > 0$, namely, $0 < \delta \leq \varepsilon/|1| = \varepsilon$ which satisfies the definition and proves that $\lim\limits_{x \to a} x =$ ________.

a

95.1 For example, if $f(x) = x$, then $\lim\limits_{x \to 4} f(x) =$ ________.

4

96. Evaluate the following limits:

(a) If $f_1(x) = 5$, then $\lim\limits_{x \to 2} f_1(x) =$ ________.

(b) If $f_2(x) = 3$, then $\lim\limits_{x \to 2} f_2(x) =$ ________.

(c) If $g_1(x) = 3x + 2$, then $\lim\limits_{x \to -1} g_1(x) =$ ______.

(d) If $g_2(x) = 2x$, then $\lim\limits_{x \to -1} g_2(x) =$ ________.

Can Theorem I be used to verify the limit in each of the above cases?

(a) 5 (b) 3 (c) -1 (d) -2 Yes - (a) and (b) are special cases of Theorem I where m = 0, and (d) is the special case where b = 0.

97. If $F(x) = f_1(x) + f_2(x) = 5 + 3 = 8$, and $G(x) = g_1(x) + g_2(x) = (3x + 2) + 2x = 5x + 2$, then:

$\lim_{x \to 2} F(x) =$ ____________.

$\lim_{x \to -1} G(x) =$ ____________.

$\lim_{x \to 2} F(x) = 8$ $\qquad\qquad$ $\lim_{x \to -1} G(x) = -3$

---

98. From frame 96, $\lim_{x \to 2} f_1(x) + \lim_{x \to 2} f_2(x) =$ ________ + ________ = ________.

Also, $\lim_{x \to -1} g_1(x) + \lim_{x \to -1} g_2(x) =$ ________ + ________ = ________.

---

5 $\quad$ 3 $\quad$ 8 $\qquad\qquad$ -1 $\quad$ -2 $\quad$ -3

---

99. Frames 97 and 98 show that for functions f_1, f_2, g_1, and g_2

$\lim_{x \to 2} (f_1(x) + f_2(x)) = \lim_{x \to 2} f_1(x) + \lim_{x \to 2} f_2(x)$ and,

$\lim_{x \to -1} (g_1(x) + g_2(x)) = \lim_{x \to -1} g_1(x) + \lim_{x \to -1} g_2(x)$

Do you think that in general it is true or false that $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$, provided both limits exist? ________

The question asked for your opinion but the statement is true and will now be stated and proved as a theorem.

---

100. Theorem II:

If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = b \pm c$.

This theorem can be stated in the following way. The limit of a sum (difference) of two functions is the sum (difference) of __________.

---

the limits of the two functions or their limits

The proof of the theorem follows in frames 101 to 105.

---

101. If $\lim_{x \to a} f(x) = b$, then $\forall\, \epsilon > 0, \exists$ a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - b| < \epsilon$. In particular, $\epsilon/2$ is greater than zero if ϵ is greater than 0, and thus, $\exists$ a δ, corresponding to $\epsilon/2$ such that $0 < |x - a| < \delta$, implies $|f(x) - b| <$ __________.

$\epsilon/2$ $\qquad\qquad$ $\forall$ means for every
$\exists$ means there exists

---

102. Similarly $\lim_{x \to a} g(x) = c$ implies that for $\varepsilon/2$ $\exists$ a $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies $|g(x) - c| <$ __________.

$\varepsilon/2$

---

103. Let δ be the smaller of δ_1 and δ_2, then both $|f(x) - b| < \varepsilon/2$ and $|g(x) - c| < \varepsilon/2$ if $0 < |x - a| <$ __________.

δ, or the smaller of $_1$ and $_2$

---

104. Consider $|(f(x) + g(x)) - (b + c)| = |(f(x) - b) + (g(x) - c)|$, and $|(f(x) - b) + (g(x) - c|$ $\underset{(\leq,\ =,\ >)}{\underline{\qquad\qquad}}$ $|f(x) - b| + |g(x) - c|$.

$\leq$, Remember the triangle inequality $|x + y| \leq |x| + |y|$.

---

105. Thus, from frames 103 and 104, if $0 < |x - a| < \delta$, then $|(f(x) + g(x)) - (b + c)| \leq |f(x) - b| + |g(x) - c| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since ε was arbitrary, it follows that $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|(f(x) + g(x) - (b + c)| <$ __________, which completes the proof of the theorem for "+".

ε

---

106. To prove the theorem in the case of the "-" sign, that is that $\lim_{x \to a}$ $(f(x) - g(x)) = b - c$, it must be shown that $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$, such that $|(f(x) - g(x)) - (b - c)| < \varepsilon$ whenever __________.

$0 < |x - a| < \delta$

---

107. Again $\lim_{x \to a} f(x) = b$ implies that $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - b| < \varepsilon$. $\varepsilon/2$ is greater than zero whenever ε is greater than zero and for $\varepsilon/2$ $\exists$ a $\delta_1 > 0$, such that $0 < |x - a| < \delta_1$ implies $|f(x) - b| <$ __________.

$\varepsilon/2$

---

108. Similarly, $\lim_{x \to a} g(x) = c$ implies for $\varepsilon/2$ $\exists$ a $\delta_2 > 0$, such that $|g(x) - c| < \varepsilon/2$ whenever $0 < |x - a| <$ ________.

δ_2

---

109. Hence, both $|f(x) - b| < \varepsilon/2$ and $|g(x) - c| < \varepsilon/2$ if δ is chosen to be ________________.

---

the smaller of δ_1 and δ_2

---

110. Now $|(f(x) - g(x)) - (b - c)| \underset{(\leq, =, \geq)}{\_\_\_\_} |(f(x) - b) - (g(x) - c)| \underset{(\leq, =, \geq)}{\_\_\_\_}$ $|f(x) - b| + |g(x) - c|$

$=$ $\leq$, remember the inequality $|x - y| \leq |x| + |y|$.

---

111. Thus, $|(f(x) - g(x)) - (b - c) \leq |f(x) - b| + |g(x) - c| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, whenever $0 < |x - a| < \delta$. Since ε is arbitrary it has been shown that $\forall \varepsilon > 0$, there exists a ______ such that $0 < |x - a| <$ ______ implies $|(f(x) - g(x) - (b - c)| < \varepsilon$. This proves that if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a}(f(x) - g(x)) = b - c$.

$\delta > 0$ δ

---

112. Let $f(x) = 2x + 4$ and $g(x) = x - 5$, then $\lim_{x \to 3} f(x) + g(x) = \lim_{x \to 3} f(x)$ + ____________ by theorem II.

$\text{LIMIT}_{x \to 3}\ g(x)$

---

113. But $\lim_{x \to 3} f(x) + \lim_{x \to 3} g(x) =$ ____________ + ____________ = ____________.

---

10 -2 8

---

114. Therefore, $\lim_{x \to 3}[(2x + 4) + (x - 5)] = 8$

For the same functions f and g, compute $\lim_{x \to 3} (f(x) - g(x)) =$ ____________.

12

115. If you were correct, go to frame 118. If you did not get 12 for the answer, go to 116.

---

116. $\lim_{x \to 3} (f(x) - g(x)) = \lim_{x \to 3} f(x) - \lim_{x \to 3} g(x)$ by the above theorem. Since $\lim_{x \to 3} f(x) = 10$ and $\lim_{x \to 3} g(x) = -2$, then $\lim_{x \to 3} f(x) - \lim_{x \to 3} g(x)$ $= 10 - (-2) =$ __________.

12

---

117. For the same functions f and g find:

(a) $\lim_{x \to -2} (f(x) + g(x)) =$ __________

(b) $\lim_{x \to 1} (f(x) - g(x)) =$ __________

-7 10

---

118. Let the product of functions f and g be denoted by $f(x)\,g(x)$, then if $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, what would you expect for $\lim_{x \to a} f(x) \cdot g(x)$?

____________________.

$\left(\lim_{x\ a} f(x)\right) \cdot \left(\lim_{x\ a} g(x)\right)$ or $b \cdot c$ is correct but this must be proven.

---

119. The theorem is stated in the following way: Theorem III:

If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a} f(x) \cdot g(x) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

$= b \cdot c$

In words this would be, the limit of the <u>product</u> of two functions is equal to the ____________________ of the functions.

---

product of the limits

---

120. To prove $\lim_{x \to a} f(x) \cdot g(x) = b\ c$ it must be shown that for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x)\ g(x) -$ ______$| <$ ____.

$b \cdot c$

---

121. If the theorem is true for all numbers ε such that $0 < \varepsilon < 1$, then it is also true for $\varepsilon > 1$ so let $0 < \varepsilon < 1$ and $0 < \varepsilon' < 1$, where ε' is a number that depends on ε.

If $\lim_{x \to a} f(x) = b$, then for ε', $\exists$ a $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies $|f(x) - b| <$ ______________.

ε'

---

122. Similarly, $\lim_{x \to a} g(x) = c$ implies that for ε', $\exists$ a $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies $|g(x) - c| <$ ______________.

ε'

---

123. If δ is the smaller of δ_1 and δ_2, then both $|f(x) - b| < \varepsilon'$ and ______________ $< \varepsilon'$.

$|g(x) - c|$

---

124. Consider $|f(x)| = |b + (f(x) - b)|$ $\underset{(\leq,\, =,\, \geq)}{\underline{\qquad\qquad}}$ $|b| + |f(x) - b|$

$\leq$

---

125. Let $0 < |x - a| < \delta$, then $|f(x) - b| < \varepsilon'$.
Adding $|b|$ to each member of $|f(x) - b| < \varepsilon'$ gives $|b| + |f(x) - b| <$
$|b| +$ ______________.

ε'

---

126. Since $\varepsilon' < 1$, $|b| + \varepsilon' < |b| + 1$. Thus if $0 < |x - a| < \delta$, results of frames 124, 125, and 126 indicate $|f(x)| <$ ______________.

$|b| + 1$

---

127. Consider:
$|f(x)g(x) - b\cdot c| = |f(x)\, g(x) - b\cdot c|$ $\underset{(\leq,=,\geq)}{\underline{\qquad\qquad}}$ $|f(x)\, g(x) - c\, f(x) + c\, f(x) - b\cdot c|$
and $|(f(x)\, g(x) - c\, f(x)) + (c\, f(x) - b\cdot c)|$
$\underset{(\leq,=,\geq)}{\underline{\qquad\qquad}}$ $|f(x)\, g(x) - c\, f(x)| + |c\, f(x) - b\cdot c|$.

$=$ $\quad\leq$ this is the triangle inequality.

---

128. If $0 < |x - a| < \delta$, then $|f(x)\, g(x) - c\, f(x)| + |c\, f(x) - b\cdot c| = |f(x)|\cdot$ $|g(x) - c| + |c||f(x) - b|$. Since $|g(x) - c| < \varepsilon'$ and $|f(x) - b| < \varepsilon'$. Then $|f(x)||g(x) - c| + |c||f(x) - b| < |f(x)$ ________ $| + |c$ ________ $|$

ε' ε'

---

129. Considering the results of frames 126 and 128, $|f(x)\, g(x) - b\cdot c| = |f(x)\, g(x) - c\, f(x) + c(f(x) - b\cdot c| \leq |f(x)\, g(x) - c\, f(x)|$ $+ |c\, f(x) - b\cdot c| = |f(x)||g(x) - c| + |c||f(x) - b| < (|b| + 1)\varepsilon' + |c|\varepsilon'$ $= (|b| + |c| + 1)$ ______________.

ε'

---

130. Remember that to prove theorem III it is necessary to find a δ corresponding to each value of ε such that $0 < |x - a| < \delta$ implies $|f(x)\, g(x) - b\cdot c| < \varepsilon$. Thus far a δ has been found such that $0 < |x - a| < \delta$ implies $|f(x)\, g(x) - b\cdot c| < (|b| + |c| + 1)\varepsilon'$. In frame 121 ε' was defined to be a number between zero and one which depends on ε. In which of the following ways can ε' be defined in terms of ε so that $|f(x)\, g(x) - b\, c| < (|b| + |c| + 1)\varepsilon'$ implies $|f(x)\, g(x) - b\cdot c| < \varepsilon$? (Circle your choice(s).

(a) $\varepsilon' = \varepsilon(|b| + |c| + 1)$ (b) $\varepsilon' = \dfrac{\varepsilon}{(|b| + |c| + 1)}$

(c) $\varepsilon' = \dfrac{\varepsilon}{(|b| + |c|)}$

(a) No, $\varepsilon' = \varepsilon(|b| + |c| + 1)$ would imply $|f(x)\, g(x) - b\, c| < \varepsilon(|b| + |c| + 1)^2$

(b) Correct, $\varepsilon' = \dfrac{\varepsilon}{|b| + |c| + 1}$ implies $|f(x)\, g(x) - b\, c| < \varepsilon$ and $\varepsilon' < 1$ since $\varepsilon < 1$ and $(|b| + |c| + 1) > 1$.

(c) No, $\varepsilon' = \dfrac{\varepsilon}{(|b| + |c|)}$ would imply $|f(x)\cdot g(x) - b\cdot c| < \dfrac{(|b| + |c| + 1)}{(|b| + |c|)}\,\varepsilon$, but $\varepsilon\dfrac{(|b| + |c| + 1)}{(|b| + |c|)}$ is greater than.

---

131. Thus, if ε' is defined to be $\dfrac{\varepsilon}{(|b| + |c| + 1)}$, then $|f(x)\, g(x) - b\cdot c| <$ $(|b| + |c| + 1)\varepsilon'$ implies $|f(x)\, g(x) - b\, c| <$ ____________. This completes the proof of the theorem.

ε

---

132. Let $f_1(x) = 3x$ and $f_2(x) = x - 2$.

$\lim\limits_{x \to 4} f_1(x) =$ ________ and $\lim\limits_{x \to 4} f_2(x) =$ ________, thus $\lim\limits_{x \to 4} f_1(x) \cdot f_2(x) = \left(\lim\limits_{x \to 4} f_1(x)\right) \cdot \left(\lim\limits_{x \to 4} f_2(x)\right) =$ ________ ________ = ________.

---

12 ; 2   12 ; 2 , 24

---

133. In a previous frame it was shown that $\lim\limits_{x \to a} mx = ma$. Consider the more general case $\lim\limits_{x \to a} K\, f(x)$, where K is a constant and $\lim\limits_{x \to a} f(x) = b$. Which of the following is the limit K f(x)? (Circle your choice(s).

(a) $K\left(\lim\limits_{x \to a} f(x)\right)$ (b) Kb

(c) $\left(\lim\limits_{x \to a} K\right)\left(\lim\limits_{x \to a} f(x)\right)$ (d) $K \cdot f(a)$

(a) Correct, the limit of a constant times a function is equal to the constant times the limit of the function. This result will be stated and proved as a theorem. Answers b and c are also correct.

(b) Correct, the limit of a constant times a function is equal to the constant times the limit of the function, and $\lim\limits_{x \to a} f(x) = b$. Answers a and c are also correct.

(c) Correct, this is a direct result of theorem III. However, since $\lim\limits_{x \to a} K = K$ answers a and b are also correct. The limit of a constant times a function is equal to the constant times the limit of the function.

(d) No, although the limit of a constant times a function is equal to the constant times the limit of the function, the $\lim\limits_{x \to a} f(x)$ is not necessarily f(a). Answers a, b, and c are correct.

---

134. Theorem IV: If $\lim\limits_{x \to a} f(x) = b$ and K is a constant, then $\lim\limits_{x \to a} K f(x) = K \lim\limits_{x \to a} f(x) = Kb$. This result can be proven directly from the definition, however, the following proof makes use of theorem III.

Let $g(x) = K$ for every x, then $\lim\limits_{x \to a} g(x) =$ ____________.

K

---

135. Therefore, $\lim K f(x) = \lim g(x) f(x) = \left(\lim\limits_{x \to a} g(x)\right)\left(\lim\limits_{x \to a} f(x)\right)$ by theorem __________.

III

---

136. Since $\lim\limits_{x \to a} g(x) = K$ and $\lim\limits_{x \to a} f(x) = b$, then $\left(\lim\limits_{x \to a} g(x)\right)\left(\lim\limits_{x \to a} f(x)\right) =$ __________ __________.

Thus, $\lim\limits_{x \to a} K f(x) = K \lim\limits_{x \to a} f(x)$, where K is constant function and the proof of theorem IV is completed.

K b

---

137. Thus, if $f(x) = 2x + 3$, $\lim\limits_{x \to 3} 4 f(x) = 4 \lim\limits_{x \to 3} f(x) = 4$ (______) = __________.

---

9 36

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138. If $f(x) = 2x - 1$ and $g(x) = 3x + 2$, find:

(a) $\lim\limits_{x \to 4} 3 \cdot f(x) =$ __________ (b) $\lim\limits_{x \to 2} 2 \cdot g(x) =$ __________

(c) $\lim\limits_{x \to 3} f(x)\, g(x) =$ __________ (d) $\lim\limits_{x \to 1} (f(x) + g(x)) =$ __________

(a) 21 (b) 16 (c) 5.11 or 55 (d) 1 + 5 or 6

---

139. Consider the functions $f_1(x) = 2x + 4$, $f_2(x) = 2$, $g_1(x) = 6x$, and $g_2(x) = 3x$.

$$\lim_{x \to 1} \frac{f_1(x)}{f_2(x)} = \lim_{x \to 1} \frac{2x + 4}{2} = \lim_{x \to 1} (x + 2) = \underline{\qquad\qquad}.$$

$$\lim_{x \to 2} \frac{g_1(x)}{g_2(x)} = \lim_{x \to 2} \frac{6x}{3x} = \lim_{x \to 2} 2 = \underline{\qquad\qquad}.$$

---

3 2

---

140. Since $\lim_{x \to 1} f_1(x) = \lim_{x \to 1} (2x + 4) = 6$ and $\lim_{x \to 1} f_2(x) = \lim_{x \to 1} 2 = 2$,

then $\dfrac{\lim_{x \to 1} f_1(x)}{\lim_{x \to 1} f_2(x)} = \dfrac{(\ \ )}{(\ \ )} =$ ________.

$\dfrac{(6)}{(3)}$; 3

---

141. For the functions f_1 and f_2 then, it is true that $\lim_{x \to 1}\left(\dfrac{f_1(x)}{f_2(x)}\right) = \dfrac{\lim_{x \to 1} f_1(x)}{\lim_{x \to 1} f_2(x)}$

Now consider $\lim_{x \to 2} g_1(x) = \lim_{x \to 2} (6x) = 12$ and $\lim_{x \to 2} g_2(x) = \lim_{x \to 2} (3x) = 6$

Then $\dfrac{\lim_{x \to 2} g_1(x)}{\lim_{x \to 2} g_2(x)} = \dfrac{\lim_{x \to 2} 6x}{\lim_{x \to 2} 3x} = \dfrac{(\ \ )}{(\ \ )} =$ ________.

$\dfrac{(12)}{(\ 6)}$ 2

---

142. Therefore, for g_1 and g_2, $\lim_{x \to 2}\left(\dfrac{g_1(x)}{g_2(x)}\right) = \dfrac{\lim_{x \to 2} g_1(x)}{\lim_{x \to 2} g_2(x)}$.

Do you suppose that in general the limit of the quotient of two functions is equal to the quotient of the limits of the functions? (Be careful.)______

The statement is not true if the limit of the denominator is zero, since this would result in an undefined form. If the limit of the numerator is also zero, an indeterminate form results. The statement is true, however, for all cases where the limit of the denominator is not zero and a proof is given next.

---

143. Theorem V: If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, $c \neq 0$

then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{b}{c}$. To prove this theorem it will first be shown that

$\lim_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$.

Theorem V then follows theorem III because $\dfrac{f(x)}{g(x)}$ 2= $f(x)$ ____________

$1/g(x)$

---

144. Let $\lim_{x \to a} g(x) = c$, to show that $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{c}$, it is required that for every $\varepsilon > 0$, $\exists$ a $\delta > 0$, such that $0 < |x - a| < \delta$ implies $|\_\_\_\_| < \_\_\_\_$.

$\left|\frac{1}{g(x)} - \frac{1}{c}\right| < \varepsilon$

---

145. Consider $\left|\frac{1}{g(x)} - \frac{1}{c}\right| = \left|\frac{c - g(x)}{g(x)\, c}\right| = \frac{|gx - c|}{|g(x)|\,|c|} = \frac{|g(x) - c|}{|c|}$ ________

$1/|g(x)|$

---

146. Since $\lim_{x \to a} g(x) = c$ it is possible to find, for any given $\varepsilon > 0$ say ε_1, a $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies $|g(x) - c| <$ ________.

ε_1

---

147. $\lim_{x \to a} g(x) \neq 0$ implies $\exists$ an interval about a in which $|g(x)| > 0$. Let $\delta_2 > 0$ be a number such that if $0 < |x - a| < \delta_2$, then $|g(x)| > 0$. In order for both $|g(x) - c| < \varepsilon$, and $|g(x)| > 0$, choose $=$ $\underset{\text{(smaller, larger)}}{\_\_\_\_\_\_\_\_}$ of δ_1, δ_2.

---

smaller, both conditions are then satisfied.

---

148. $|g(x)| > 0$ implies a number d such that $|g(x)| > d > 0$. Thus,

$\frac{1}{|g(x)|} \underset{(<,\ =,\ >)}{\_\_\_\_} \frac{1}{d}$ when $0 < |x - a| < \delta$

---

149. In frame 146 it was shown that $|g(x) - c| < \varepsilon_1$, whenever $0 < |x - a| < \delta_1$. But $|g(x) - c| < \varepsilon_1$ implies $\frac{|g(x) - c|}{|c|} < \frac{\varepsilon_1}{\_\_\_\_}$

$|c|$

149.1 $\frac{|g(x) - c|}{|c|} < \frac{\varepsilon_1}{|c|}$ and $\frac{1}{|g(x)|} < \frac{1}{d}$ implies

$\frac{|g(x) - c|}{|c|} \cdot \frac{1}{|g(x)|} < \frac{\varepsilon_1}{|c|} \cdot$ ________

$1/d$

---

150. Let $\epsilon_1 < d \cdot |c|\, \epsilon$ so that $\frac{\epsilon_1}{d\,|c|} < \epsilon$ then putting together the results of frame 145-150, $\left|\frac{1}{g(x)} - \frac{1}{c}\right| = \frac{|g(x) - c|}{|c|} \cdot \frac{1}{|g(x)|} < \frac{\epsilon_1}{d\,|c|} <$ __________,
whenever $0 < |x - a| < \delta$.

ϵ

---

151. Therefore, $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{c}$ provided that $\lim_{x \to a} g(x) = c \neq 0$. Now to complete the proof of theorem V, consider $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \cdot \frac{1}{g(x)}$

$= \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = b \cdot$ __________ = __________.

$\frac{1}{c}$ $\frac{b}{c}$

---

152. Let $f_1(x) = 3x + 5$, $f_2(x) = x - 3$, and $f_3(x) = 2x + 1$. Find:

(a) $\lim_{x \to 2} \frac{f_1(x)}{f_2(x)} =$ __________

(b) $\lim_{x \to 1} \frac{f_3(x)}{f_1(x)}$

(c) $\lim_{x \to 3} \frac{f_3(x)}{f_2(x)} =$ __________

(a) $\frac{11}{-1}$ or -11

(b) 3/8

(c) The theorem does not apply since the $\lim_{x \to 3} f_2(x) = 0$. The limit does not exist in the sense of the definition of this program.

---

153. The reason that part (c) above does not follow from theorem V is because the theorem requires the limit of the function in the denominator to be

____

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non-zero or $\neq 0$

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154. The theorems proven thus far apply to any algebraic functions whose limits exist. However, examples have involved only linear functions, i.e. functions of the form $f(x) = mx + b$. Consider now a more general type function, the real polynomial function. A real polynomial function is a function of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \text{----} + a_{n-1}x + a_n$. where n is a positive integer, x is a real number and a_i, i= 0, ----, n are real constants.

Is the function $f(x) = 3x + 4$ a polynomial function? __________

Yes, for this function $a_0 = 3$, $n = 1$, and $a_1 = 4$.

---

155. $f(x) = 3x + 4$ is also a linear function. Every linear function is a polynomial function. Every polynomial function $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ a linear function.

---

is not, the function $2x^2 + 3x + 1$ is a polynomial function but it is not a linear function.

---

156. What would you guess is the limit of a polynomial function such as $f(x) = c_0x^n + c_1x^{n-1} + \text{---------} + c_n$?

$\lim\limits_{x \to a} (c_0x^n + c_1x^{n-1} + \text{--------} + c_n) =$ ______________

$c_0a^n + c_1a^{n-1} + \text{---------} + c_{n-1}\,a + c_n$ is the correct answer.

---

157. Theorem VI: $\lim\limits_{x \to a} c_0x^n + c_1x^{n-1} + \text{----} + c_n = c_0a^n + c_1a^{n-1} + \text{----} + c_n.$

Your knowledge of limits of linear functions may have led you to the correct answer in 156. Consider now what might be done to prove the above result. It has already been shown that $\lim\limits_{x \to a} c\, f(x) = c\,.$______________

$\lim\limits_{x \to a} f(x)$

---

158. It must yet be proven that $\lim\limits_{x \to a} x^n = a^n$ and that $\lim\limits_{x\quad a} (f_1(x) + f_2(x) + \text{----} + f_n(x)) = \lim\limits_{x \to a} f_1(x) + \lim\limits_{x \to a} f_2(x) + \text{----} + \lim\limits_{x \to a} f_n(x).$ Combining these three results, it can be shown that $\lim\limits_{x \to a} c_0x^n + c_1x^{n-1} + \text{----} + c_{n-1}x + c_n =$ ______________.

$c_0a^n + c_1a^{n-1} + \text{----} + c_n$

---

159. Theorem VII: $\underset{x \to a}{\text{limit}}\ f_1(x) + \_\_\_\_ + f_n(x) = \underset{x \to a}{\text{limit}}\ f_1(x) + \_\_\_\_ + \underset{x \to a}{\text{limit}}\ f_n(x)$, provided each of the limits exist. To prove theorem VII, mathematical induction will be used. It has been proven, theorem II, that $\underset{x \to a}{\text{limit}}\ (f_1(x) - f_2(x)) = \underset{x \to a}{\text{limit}}\ f_1(x) + \underset{x \to a}{\text{limit}}\ f_2(x)$. This is the first step of a mathematical induction proof. That is, the theorem is true for $n = 2$. (the theorem is trivally true for $n = 1$). The next step is to assume that the theorem is true for the case $n - 1$. That is $\underset{x \to a}{\text{limit}}\ (f_1(x) + \_\_\_ + f_{n-1}(x)) = \underset{x \to a}{\text{limit}}\ f_1(x) + \_\_\_$ ______________.

$\underset{x \to a}{\text{LIMIT}}\ f_{n-1}(x)$

---

160. It must be shown next if the theorem is true for $n - 1$, then it is true for ______________.

---

n

---

161. Consider $f_1(x) + f_2(x) + \_\_\_\_ + f_n(x) = (f_1(x) + \_\_\_\_ + f_{n-1}(x)) + f_n(x)$. Let $f_1(x) + \_\_\_\_ + f_{n-1}(x) = h(x)$. Then $\underset{x \to a}{\text{limit}}\ (f_1(x) + \_\_\_\_ + f_n(x)) = \underset{x \to a}{\text{limit}}\ (h(x) + f_n(x)) =$ ______________ by Theorem II. (See Panal).

$\underset{x \to a}{\text{limit}}\ h(x) + \underset{x \to a}{\text{limit}}\ f_n(x)$

---

162. This is the result of theorem II:

But $\underset{x \to a}{\text{limit}}\ h(x) = \underset{x \to a}{\text{limit}}\ (f_1(x) + f_2(x) + \_\_\_\_ + f_{n-1}(x))$

$= \underset{x \to a}{\text{limit}}\ (f_1(x) + \underset{x \to a}{\text{limit}}\ f_2(x) + \cdots +$ __________

by the induction hypothesis. (i.e. the assumption that the theorem is true for n-1)

$\underset{x \to a}{\text{limit}}\ f_{n-1}(x)$

---

163. Thus, $\lim_{x \to a} (f_1(x) + \text{----} + f_n(x)) = \lim_{x \to a} (h(x) + f_n(x))$

$= \lim_{x \to a} h(x) + \lim_{x \to a} f_n(x)$

$= \lim_{x \to a} f_1(x) + \ldots + \lim_{x \to a} f_{n-1}(x) +$ _____

And this completes the proof of theorem VII.

---

164. The next step in verifying the limit of a polynomial function is to prove that $\lim_{x \to a} x^n =$ __________.

---

a^n

---

165. Consider the following general result. Theorem VIII: $\lim_{x \to a} f^n(x) = \left(\lim_{x \to a} f(x)\right)^n$, provided $\lim_{x \to a} f(x)$ exists and n is a positive integer. The desired result $\lim_{x \to a} x^n = a^n$ could be obtained as a special case of Theorem VIII by letting $f(x) =$ ________.

---

x

---

166. To prove Theorem VIII, $\lim_{x \to a} f^n(x) = \lim_{x \to a} f(x)^{\ n}$, mathematical induction will again be used. It has been shown, Theorem III, that $\lim_{x \to a} f_1(x) \cdot f_2(x) = \left(\lim_{x \to a} f_1(x)\right)\left(\lim_{x \to a} f_2(x)\right)$. If $f(x) = f_1(x) = f_2(x)$, then the statement becomes $\lim_{x \to a} f^2(x) = \lim_{x \to a} f(x) \quad \lim_{x \to a} f(x) =$ ________________. This is step 1 of the mathematical induction proof.

$\lim_{x \to a} f(x)^{\ 2}$

---

167. According to mathematical induction procedures, it must now be assumed that $\lim_{x \to a} f^{n-1}(x) =$ ______________.

$\lim_{x \to a} f(x)^{\ n-1}$

---

168. If $\lim_{x \to a} f(x)^{n-1} = \left(\lim_{x \to a} f(x)\right)^{n-1}$, then consider $f^n(x) = f^{n-1}(x) \cdot f(x)$.
Let $h(x) = f^{n-1}(x)$, then $f^n(x) =$ __________ $\cdot f(x)$.

$h(x)$

169. Now $\lim_{x \to a} f^n(x) = \left(\_\_\_\_\_\_\_\_\_\_ \right)\left(\lim_{x \to a} f(x)\right)$ because of Theorem III.

$\lim_{x \to a} h(x)$

170. But $\lim_{x \to a} h(x) = \lim_{x \to a} f^{n-1}(x) = \left(\_\_\_\_\_\_\_\_\_\_ \right)$ by the induction hypothesis.

$\lim_{x \to a} f(x)^{n-1}$

171. Therefore, $\lim_{x \to a} f^n(x) = \lim_{x \to a} (h(x) \cdot f(x)) = \left(\lim_{x \to a} h(x)\right)\left(\lim_{x \to a} f(x)\right) =$
$\left(\lim_{x \to a} f^{n-1}(x)\right)\left(\lim_{x \to a} f(x)\right) = \left(\lim_{x \to a} f(x)\right)^{n-1} \cdot \left(\lim_{x \to a} f(x)\right)$
$=$ __________

And this completes the proof of the theorem.

$\left(\text{LIMIT}_{x \to a} f(x)\right)^n$

172. The proof of the fact that $\lim_{x \to a} c_0x^n + \text{----} + c_n = c_0a^n + \text{----} + c_n$,
Theorem VI can now be completed. Consider

$\lim_{x \to a} (c_0x^n + \text{----} + c_{n-1}x + c_n) = \lim_{x \to a} c_0x^n + \text{----} +$ __________ $+ \lim_{x \to a} c_n$,
by theorem VII.

$\lim_{x \to a} c_{n-1} \cdot x$

173. But for each i, $i = 0, \text{----}, n-1$, $\lim_{x \to a} c_i x^{n-i} = c_i \cdot$ __________ by
Theorem IV. And $\lim_{x \to a} c_n =$ __________ by theorem I. (See Panel)

$\lim_{x \to a} x^{n-i}$; c_n

174. Also for each i, i = 0, 1, ..., n-1, $\lim\limits_{x \to a} x^{n-i}$ = ________ by Theorem VIII.

= ________ by Theorem I.

$\left(\lim\limits_{x \to a} x\right)^{n-i}$ a^{n-i}

---

175. Putting together the results of frames 172-174, $\lim\limits_{x \to a} c_0x^n + \ldots + c_{n-1}x + c_n = c_0\left(\lim\limits_{x \to a} x^n\right) + c_1\left(\lim\limits_{x \to a} x^{n-1}\right) + \ldots + c_{n-1}\left(\lim\limits_{x \to a} x\right) + \lim\limits_{x \to a} c_n$

$= c_0a^n + c_1a^{n-1} + \ldots + c_{n-1} \cdot$ ____________ + ____________, and the proof is complete.

---

a c_n

---

176. Evaluate the following limits:

(a) $\lim\limits_{x \to -1} (x^3 + 2x^2 - 3x - 4)$ = ____________

(b) $\lim\limits_{x \to a} (3x^4 + 2x + 1)$ = ____________

(c) $\lim\limits_{x \to 1} (5x^3 + 6x^2 + 3x + 4)$ = ____________

---

0 53 18

---

177. If $f(x)$ and $g(x)$ are polynomial functions, then the function $h(x) = \frac{f(x)}{g(x)}$, is a rational function whose domain is $\{x \mid g(x) \neq 0\}$. Which of the following are rational functions? (Circle correct choice(s)).

(a) $f_1(x) = \frac{3x^2 + 2x + 1}{2x^2 + x + 5}$ (b) $f_2(x) = \frac{x^2 + 2x - 3}{x - 1}$

(c) $f_3(x) = \frac{2x + 1}{x^2 + 1}$ (d) $f_4(x) = \sin x$

(a) $f_1(x) = \frac{3x^2 + 2x + 1}{2x^2 + x + 5}$ is a rational function.

(b) $f_2(x) = \frac{x^2 + 2x - 3}{x - 1}$ is a rational function.

(c) $f_3(x) = \frac{2x + 1}{x^2 + 1}$ is not a rational function.

(d) $f_4(x) = \sin x$ is not a rational function. In fact it is a not algebraic or transcendental function.

178. If $f(x) = \dfrac{3x^2 + 4x + 1}{2x^3 + 4x^2 + 3x + 5}$, what would you guess for

$\lim\limits_{x \to 2} f(x)$? i.e. $\lim\limits_{x \to 2} \dfrac{3x^2 + 4x + 1}{2x^3 + 4x^2 + 3x + 5}$ = ______________

$\frac{21}{43}$ is correct. See explanation below.

---

179. A rational function is the quotient of two polynomial functions and thus can usually be evaluated by first applying theorem V and then theorem VI.

For instance in the above example, $\lim\limits_{x \to 2} \dfrac{3x^2 + 4x + 1}{2x^3+4x^2+3x+5} = \dfrac{\lim\limits_{x \to 2} (3x^2 + 4x + 1)}{\lim\limits_{x \to 2} (2x^3+4x^2+3x+5)}$

by Theorem V and

$\dfrac{\lim\limits_{x \to 2}(3x^2 + 4x + 1)}{\lim\limits_{x \to 2}(2x^3 + 4x^2 + 3x + 5)}$ = ______________ according to Theorem VI.

$\frac{21}{43}$

---

180. Evaluate the limits of the following rational functions:

(a) $f_1(x) = \dfrac{3x^2 + 2x + 1}{2x^2 + x + 5}$, $\lim\limits_{x \to 1} f_1(x)$ = ______________

(b) $f_2(x) = \dfrac{4x + 1}{x^2 + 3}$, $\lim\limits_{x \to 2} f_2(x)$ = ______________

(c) $f_3(x) = \dfrac{x^2 + 2x - 3}{x - 1}$, $\lim\limits_{x \to 2} f_3(x)$ = ______________

6/8 or 3/4   9/7   5/1 or 5

---

181. Suppose that it is desired to find $\lim\limits_{x \to 1} f_3(x)$ where $f_3(x) = \dfrac{x^2 + 2x - 3}{x - 1}$

Is the method used in frame 180 applicable in this case? ______________

---

no, $\lim\limits_{x \to 1} (x - 1) = 0$, and theorem V does not hold if the limit of the denominator is equal to zero.

---

182. In determining $\lim_{x \to a} f(x)$, x __________ (does, does not) actually equal a.

---

does not

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183. If $x \neq a$, then for all other values of x, $\frac{(x - b)(x - a)}{(x - a)(x - d)} =$ __________.

$\frac{x - b}{x - d}$

---

184. Thus, $\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} = \lim_{x \to 1} \frac{(x + 3)(x - 1)}{(x - 1)}$

$= \lim_{x \to 1} (x + 3) =$ __________.

4

---

185. It can now be proven that the procedure used above is valid.

Theorem IX: Let f(x) and g(x) be functions such that f(x) = g(x) for every x, except x = a. If $\lim_{x \to a} g(x)$ exists then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$. For example, in frame 184 $f(x) = \frac{x^2 + 2x - 3}{x - 1}$ and $g(x) = x + 3$, then f(x) = g(x), except at x = __________, but $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = 4$.

1

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186. Proof of theorem IX.

By hypothesis $\{x \mid x \in \text{domain } f \text{ and } x \neq a\} = \{x \mid x \in \text{domain } g \text{ and } x \neq a\}$. Because of the set equality this is one set described in two ways. Denote this set by X. Then for every $x \in X$, which of the following are true? (Circle correct choice(s).)

(a) f(x) = g(x)

(b) f(x) - b = g(x) - b

(c) f(x) - b = g(x) - d, $b \neq d$

(a) correct, by hypothesis of theorem. b is also correct.

(b) correct, since f(x) = g(x) for $x \neq a$ and the same number b is subtracted from both members of the equality. a is also correct.

(c) No, since f(x) = g(x) for $x \neq a$, then subtracting different numbers would cause the result to be unequal. a and b are correct.

---

187. Let $\lim_{x \to a} g(x) = b$, then by definition, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $0 < |x - a| < \delta$ implies ____________.

$|g(x) - b| < \varepsilon$

---

188. Now $\{x \mid 0 < x - a < \delta,\ x \in \text{domain } g\} \subset \{x \mid x \in \text{domain } g \text{ and } x \neq a\} = X$.

Thus, $\{x \mid 0 < |x - a| < \delta,\ x \in \text{domain } g\} \subset$ ____________.

X

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189. Therefore, elements in the set $\{x \mid 0 < x - a < \delta,\ x \in \text{domain } g\}$ are also elements in the domain of f and by frame 186 for the values of x, $f(x) - b = g(x) - b$. Hence, $|f(x) - b| = |$__________$|$.

$|g(x) - b|$

---

190. Also by frame 187 for these values of x, $|g(x) - b| < \varepsilon$. Therefore $\forall\ \varepsilon > 0$, for all $x \in$ domain f such that $0 < |x - a| < \delta$, it is true that $|f(x) - b| = |g(x) - b|$ and thus $|f(x) - b| <$ ____________.

ε

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191. It has thus been shown that $\forall\ \varepsilon > 0$, $\exists$ a $\delta > 0$ such that $0 < |x - a| < \delta$, $x \in$ domain f, implies $|f(x) - b| < \varepsilon$. Thus, $\lim_{x \to a} f(x) =$ ____________ and so $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ which completes the proof of the theorem.

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b

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192. Let $f(x) = \dfrac{x^2 + 3x + 2}{x + 1}$.

Then, $f(x) = \dfrac{(x + 2)(x + 1)}{x + 1} = x + 2$ if $x \neq$ ____________

thus, $\lim_{x \to -1} \dfrac{(x + 2)(x + 1)}{(x + 1)} = \lim_{x \to -1}$ ____________ by Theorem IX. Therefore,

$\lim_{x \to -1} f(x) = \lim_{x \to -1} (x + 2) =$ ____________ by theorem __________.

-1 $(x + 2)$ +1   I

---

193. Let $f(x) = \dfrac{x^2 + x - 6}{x^2 - x - 2} = \dfrac{(x - 2)(x + 3)}{(x + 1)(x - 2)}$

then $f(x) = \dfrac{x + 3}{x + 1}$ if $x \neq$ ________, $\lim\limits_{x \to 2} \dfrac{(x - 2)(x + 3)}{(x + 1)(x - 2)} = \lim\limits_{x \to 2}$ ________
by Theorem IX

then $\lim\limits_{x \to 2} f(x) = \dfrac{\lim\limits_{x \to 2} (x + 3)}{\lim\limits_{x \to 2} (x + 1)}$ by theorem ________.

thus $\lim\limits_{x \to 2} f(x) =$ ________ by theorem ________.

---

2 $\quad \dfrac{x + 3}{x + 1} \quad$ V $\quad$ 5/3 $\quad$ I

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194. The above examples were picked so that after application of one or both of theorems V and IX, a limit existed. There are cases, however, where either the limit does not exist, the limit is $\pm\infty$ which is not a real number and hence not considered at this time, or a limit exists, but the method of finding it is beyond the scope of this program. Which of the following limits can be found by the methods of this program? (Circle correct choice(s)).

(a) $\lim\limits_{x \to 2} \dfrac{x + 2}{x - 2}$ $\qquad$ (b) $\lim\limits_{x \to 0} \dfrac{x}{(2^x - 1)}$

(c) $\lim\limits_{x \to \infty} \dfrac{x}{x^2 + 5}$ $\qquad$ (d) $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 4}$

(a) No, $\lim\limits_{x \to 2} (x - 2) = 0$ and theorem V does not apply. Since there does not exist a function whose $\lim\limits_{x \to 2}$ exists, that is equal to $\dfrac{x + 2}{x - 2}$ for all x except $x = 2$.

Theorem IX cannot be employed. This is an example of a function having an infinite limit at a point which is not dealt with in this program.

(b) No, the limit of both numerator and denominator is zero, an indeterminate form. Theorems V and IX do not apply. Methods of finding the limit are beyond the scope of this program.

(c) No, ∞ is not a real number and methods of finding limits as x approaches ∞ are not dealt with in this program.

(d) Yes, the limit is 1/4, use theorems IX and V.

---

195. Another limit theorem is necessary to evaluate and verify the limits of functions which involve the nth roots of expressions.

Theorem X: If $\lim\limits_{x \to a} f(x) = b$, then $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} = \sqrt[n]{b}$, where n is a positive integer and n is odd if $\lim\limits_{x \to a} f(x) < 0$.

The proof of this theorem is omitted since it involves concepts beyond the scope of this program.

Consider $f(x) = x^2 + 1$, $\lim\limits_{x \to 2} \sqrt[3]{f(x)} = \sqrt[3]{\lim\limits_{x \to 2} f(x)}$ by theorem X

$= \sqrt[3]{\_\_\_\_\_\_\_\_}$ by theorem _____.

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5 VI

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196. Let $f(x) = 2x^2 - 5$.

$\lim\limits_{x \to 1} \sqrt[3]{f(x)} = \sqrt[3]{\_\_\_\_\_\_\_\_}$ by theorem ________.

$= \sqrt[3]{\_\_\_\_\_\_\_\_}$ by theorem ________.

$\lim\limits_{x\;1} f(x)$ or $\lim\limits_{x\;1} (2x^2 - 5)$ X -3 VI

---

197. The real functions considered to this point have been special cases of the set of real algebraic functions. The real algebraic function consist of a finite number of operations of addition, subtraction, multiplication, division, and finite powers and roots.

Which of the following are algebraic function?

(a) $f(x) = \dfrac{x - 2}{\sqrt{x^2 - 4}}$ (b) $f(x) = \sqrt[5]{\dfrac{x^3 + 14}{x^4 - 15}}$ (c) $f(x) = x$

(d) $f(x) = 2^x$ (e) $f(x) = \tan x$ (f) $f(x) = \dfrac{(x - 1)^3}{\sqrt{x^2 + 3} - 2}$

Answers are on next page

(a) Yes, $f(x) = \dfrac{x - 2}{\sqrt{x^2 - 4}}$ is an algebraic function.

(b) Yes, $f(x) = \dfrac{1}{2\sqrt{x - 4}}$ is an algebraic function.

(c) Yes, $f(x) = x$ is an algebraic function.

(d) No, $f(x) = 2^x$ is an exponential function.

(e) No, $f(x) = \tan x$ is a transcendental function.

(f) Yes, $f(x) = \dfrac{x - 1}{\sqrt{x^2 + 3} - 2}$ is an algebraic function.

---

198. Consider $f(x) = \dfrac{x - 2}{\sqrt{x^2 - 4}}$

$\lim\limits_{x \to 3} \dfrac{x - 2}{\sqrt{x^2 - 4}} = \dfrac{\_\_\_}{\lim\limits_{x \to 3} \sqrt{x^2 - 4}}$ by theorem __________.

$\lim\limits_{x \to 3} (x - 2)$ V

---

199. $\dfrac{\lim\limits_{x \to 3} (x - 2)}{\lim\limits_{x \to 3} \sqrt{(x^2 - 4)}} = \dfrac{\lim\limits_{x \to 3} (x - 2)}{\sqrt{\_\_\_\_\_\_\_\_}}$ by theorem X

$\lim\limits_{x \to 3} (x^2 - 4)$

---

200. $\dfrac{\lim\limits_{x \to 3} (x - 2)}{\sqrt{\lim\limits_{x \to 3} (x^2 - 4)}} = \dfrac{1}{\sqrt{}}$ by theorems I and ________.

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5 VI

201. Let $f(x) = \sqrt[5]{\dfrac{x + 4}{x^2 + 2x + 4}}$

$$\lim_{x \to 2} \sqrt[5]{\frac{x + 4}{x^2 + 2x + 4}} = \sqrt[5]{\lim_{x \to 2} \frac{x + 4}{x + 2x + 4}}$$ by theorem ________.

X

---

202. $$\sqrt[5]{\lim_{x \to 2}\left(\frac{x + 4}{x^2 + 2x + 4}\right)} = \sqrt[5]{\frac{\lim_{x \to 2} (x + 4)}{\lim_{x \to 2} (x^2 + 2x + 4)}}$$ by theorem ________.

V

---

203. $$\sqrt[5]{\frac{\lim_{x \to 2} (x + 4)}{\lim_{x \to 2} (x^2 + 2x + 4)}} = \sqrt[5]{\rule{3em}{0.4pt}}$$ by theorems ________ and ________.

6/12 or 1/2 I VI

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204. Let $f(x) = \dfrac{4 + x - 2}{x}$, then $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\dfrac{\sqrt{4 + x} - 2}{x}\right)$.

Since evaluation by direct application of the limit theorems would result in an indeterminate form, consider the following.

$$\frac{\sqrt{4 + x} - 2}{x} = \frac{(\sqrt{4 + x} - 2)(\sqrt{4 + x} + 2)}{x(\sqrt{4 + x} + 2)} = \frac{\rule{3em}{0.4pt}}{x(\sqrt{4 + x} + 2)}$$

---

4 + x - 4 or x

---

205. $$\frac{4 + x - 4}{x(\sqrt{4 + x} + 2)} = \frac{x}{x(\sqrt{4 + x} + 2)} = \frac{\rule{3em}{0.4pt}}{(\sqrt{4 + x} + 2)}$$

1

---

206. Since $\frac{\sqrt{4+x}-2}{x} = \frac{1}{\sqrt{4+x}+2}$ except at $x = 0$, then

$$\lim_{x \to 0}\left(\frac{\sqrt{4+x}-2}{x}\right) = \lim_{x \to 0}\left(\frac{1}{\sqrt{4+x}+2}\right)$$ by theorem ________.

IX

---

207. $$\lim_{x \to 0}\left(\frac{1}{\sqrt{4+x}+2}\right) = \frac{1}{\lim_{x \to 0}(\sqrt{4+x}+2)}$$ by theorem V

and $$\frac{1}{\lim_{x \to 0}(\sqrt{4+x}+2)} \quad \frac{1}{\_\_\_\_ + \_\_\_\_}$$ by theorem II.

$\lim_{x \to 0}\sqrt{4+x}$ $\qquad$ $\lim_{x \to 0} 2$

---

208. $$\frac{1}{\lim_{x \to 0}\sqrt{4+x} + \lim_{x \to 0} 2} = \frac{1}{\_\_\_\_ + \lim_{x \to 0} 2}$$ by theorem X.

$\sqrt{\lim_{x \to 0}(4+x)}$

---

209. $$\frac{1}{\sqrt{\lim_{x \to 0}(4+x)} + \lim_{x \to 0} 2} = \frac{1}{\sqrt{\_\_\_\_} + \_\_\_\_} = \_\_\_\_$$ by theorem _____.

Therefore, the $\lim_{x \to 0}\frac{\sqrt{4+x}-2}{x} = \frac{1}{4}$

---

4 $\qquad$ 2 $\qquad$ 4 $\qquad$ I

---

210. Using the methods demonstrated in this program, evaluate the following limits:

(a) $f(x) = \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$, $\lim_{x \to 1} f(x) =$ ________

(b) $f(x) = \frac{\sqrt{x^2+1}-1}{x}$, $\lim_{x \to 0} f(x) =$ ________.

(c) $f(x) = \frac{x^2 - 1}{x - 1}$, $\lim_{x \to 1} f(x) =$ ________.

(a) 1/2 $\qquad$ (b) 0 $\qquad$ (c) 2

LIST OF THEOREMS

Theorem I: If $f(x) = mx + b$, where m and b are real constants, then $\lim_{x \to a} (mx + b) = ma + b$.

Theorem II: If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = b \pm c$.

Theorem III: If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a} f(x) \cdot g(x)$

$$= \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = b \cdot c$$

Theorem IV: If $\lim_{x \to a} f(x) = b$ and K is a constant, then $\lim_{x \to a} K\,f(x) = K\left(\lim_{x \to a} f(x)\right) = K \cdot b$

Theorem V: If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, $c \neq 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} =$

$$\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{b}{c}$$

Theorem VI: $\lim_{x \to a} (c_0 x^n + c_1 x^{n-1} + \text{----} + c_n) = c_0 a^n + c_1 a^{n-1} + \text{----} + c_n$.

Theorem VII: $\lim_{x \to a} f_1(x) + \text{----} + f_n(x) = \lim_{x \to a} f_1(x) + \text{----} + \lim_{x \to a} f_n(x)$, provided each of the limits exist.

Theorem VIII: $\lim_{x \to a} f^n(x) = \lim_{x \to a} f(x)^n$, provided $\lim_{x \to a} f(x)$ exists and n is a positive integer.

Theorem IX: Let $f(x)$ and $g(x)$ be functions such that $f(x) = g(x)$ for every x, except $x = a$. If $\lim_{x \to a} g(x)$ exists then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

Theorem X: If $\lim_{x \to a} f(x) = b$, then $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{b}$, where n is a positive integer and n is odd if $\lim_{x \to a} f(x) < 0$.

APPENDIX C

TESTS OF THE LEARNING SETS

VIIA. Absolute value concept.

1. The definition of $|x|$ is:

(a) $|x| = x$ (b) $|x| = \begin{cases} -x \text{ if } x \text{ is positive} \\ 0 \text{ if } x \text{ is negative} \end{cases}$

(c) $|x| = \begin{cases} -x \text{ if } x \text{ is positive} \\ 0 \text{ if } x \text{ is zero} \\ x \text{ if } x \text{ is negative} \end{cases}$ (d) $|x| = \begin{cases} x \text{ if } x \text{ is positive} \\ 0 \text{ if } x \text{ is zero} \\ -x \text{ if } x \text{ is negative} \end{cases}$

2. $|15| =$ ____________

(a) 0 (b) 15 (c) -15 (d) $\frac{1}{15}$

3. $|a \cdot b| =$ ____________

(a) $a \cdot b$ (b) $\frac{|a|}{|b|}$

(c) $|a| \cdot |b|$ (d) $|a| + |b|$

4. $|7 - 4| + |8 - 12| =$ ____________

5. $|a + b|$ ________ $|a| + |b|$

(a) $\leq$ (b) = (c) $\geq$ (d) no correct answer

VIIB. Algebra of inequalities.

1. $a > b$ if and only if:

(a) $b - a$ is positive (b) $a - b$ is positive

(c) $a - b$ is negative (d) $a \neq b$

2. $a < b$ if and only if $a + c$ ________ $b + c$

(a) $<$ (b) = (c) $>$ (d) no correct answer

3. If $a > b$ and $c < 0$, then ac _______ bc

(a) $<$ (b) $=$ (c) $>$ (d) no correct answer

4. $a < b$ if and only if $\frac{1}{a}$ _________ $\frac{1}{b}$

5. For which of the following cases is $a \cdot b > 0$

(a) $a < 0,\ b > 0$ (b) $a > 0,\ b = 0$ (c) $a < 0,\ b < 0$

(d) $a = 0,\ b > 0$

VIIC. Algebra of functions

1. If $f(x) = 3x$, then $f(2) =$ _______________.

2. If $f(x) = 5x$ and $g(x) = 2x + 1$, then $f(x) + g(x) =$ _______.

3. If $f(x) = 2x - 5$ and $g(x) = x - 5$ then $f(3) - g(2) =$ _______________.

4. If $f(x) = 6x - 1$ and $g(x) = 3x$ then $f(x)g(x) =$ ___________.

5. If $f(x) = x^2 + 2x + 5$ and $g(x) = x^2 - x + 3$, then $f(3)/g(5) =$ _______________________.

VI. Discovery of limits intuitively

1. If $\lim_{x \to 2} 5x = 10$, then 5x approaches 10 as x approaches ____.

2. $\lim_{x \to 3} 2x =$ __________.

3. $\lim_{x \to 1} 4x + 2 =$ __________.

4. $\lim_{x \to 2} \frac{x + 2}{x + 3} =$ _____________.

5. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} =$ ______________.

V. Definition of limit of a function

______1. The definition of the statement $\lim_{x \to a} f(x) = b$ means:

(a) If there exists an $\varepsilon > 0$ such that for every $\delta > 0$,

$0 < |x - a| < \delta$ implies $|f(x) - b| < \varepsilon$.

(b) If for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f(x) - b| < \varepsilon$ implies $0 < |x - a| < \delta$.

(c) If for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$0 < |x - a| < \delta$ implies $|f(x) - b| < \varepsilon$.

(d) If there exists a $\delta > 0$ such that for every $\varepsilon > 0$,

$|f(x) - b| < \delta$ implies $0 < |x - a| < \varepsilon$.

2. If $f(x) = 2x$ then $0 < |x - 3| < \frac{1}{2}$ implies $|f(x) - 6| <$ ____

(a) 1/2 (b) 3/4 (c) 7/8 (d) 1

3. If $f(x) = 3x$, then $0 < |x - 2| < \frac{1}{9}$ implies $|f(x) - 6| <$ ____

(a) 1/8 (b) 3 (c) 1/6 (d) 1/27

4. If $f(x) = 4x$, then $0 < |x - 7| < \frac{1}{4}$ implies $|f(x) - 14| <$ ____

5. Given $f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}$, is $\varepsilon = 2$ a value of ε that

could be used to demonstrate that $\lim_{x \to 0} f(x) \neq 0$? ____

IVA. Evaluation and verification of the limit of linear functions.

1. Theorem I states: If $f(x) = mx + b$, where m and b are real constants, then $\lim_{x \to a} (mx + b) =$ ____

(a) m + ab

(b) ma + b

(c) am

(d) ba + m

2. $\lim_{x \to 2} (2x + 1) =$ ____

(a) 4 (b) 5 (c) 2(2) - 1 (d) 3

3. $\lim_{x \to 3} 7x =$ ____

4. $\lim_{x \to 3} (13x - 3)$ ________

5. If $0 < |x - a| < \delta$ implies $|(mx + b) - (ma + b)| < \varepsilon$ then $0 < \delta \leq$ ________.

IVB. Evaluation and verification of the limit of the sum and difference of functions.

1. If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, then $\lim_{x \to a} \big(f(x) + g(x)\big) =$ ________.

   (a) b - c (b) b · c (c) c - b (d) b + c

2. If $f(x) = x$ and $g(x) = 3x$, then $\lim_{x \to 2} \big(f(x) + g(x)\big) =$ ________

   (a) 4 (b) 8 (c) 12 (d) 16

3. If $f_1(x) = 4x$ and $f_2(x) = 7x$, then $\lim_{x \to 1} \big(f_1(x) + f_2(x)\big) =$ ________

4. If $g(x) = 5x$ and $h(x) = 2x + 3$, then $\lim_{x \to 3} \big(g(x) - h(x)\big) =$ ________

5. If $f(x) = 3x - 2$ and $h(x) = 4x + 1$, then $\lim_{x \to 4} \big(h(x) - f(x)\big) =$ ________

IVC. Evaluation and verification of the limit of the product of functions.

1. If the $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$ then $\lim_{x \to a} f(x) \cdot g(x) =$ ________

   (a) b·c (b) b/c (c) b + c (d) b - c

2. If $f(x) = 7x$ and $g(x) = 3x$ then $\lim_{x \to 2} f(x) \cdot g(x) =$ ________

   (a) 21 (b) 42 (c) 63 (d) 84

3. If $f(x) = 3x - 7$, then $\lim_{x \to 4} 10 \cdot f(x) =$ ________

4. If $h_1(x) = 6x + 5$ and $h_2(x) = 2x - 1$, then $\lim_{x \to 2} h_1(x) \cdot h_2(x) =$ ________

5. If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$ then to prove $\lim_{x \to a} f(x)g(x) = b \cdot c$ it must be shown that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies ________

IVD. Evaluation and verification of the limit of the quotient of functions.

1. If $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$ then $\lim_{x \to a} f(x)/g(x) =$ ________

   (a) b+ c (b) b - c (c) b·c (d) b/c

2. If $f(x) = 5x$ and $g(x) = 4x$, then $\lim_{x \to 2} f(x)/g(x) =$ ________

   (a) 4/5 (b) 5/4 (c) 8/10 (d) 10/4

3. If $f_1(x) = 3$ and $f_2(x) = x + 2$, then $\lim_{x \to 4} \frac{f_1(x)}{f_2(x)} =$ ________

4. If $g_1(x) = 4x + 3$ and $g_2(x) = 3x - 1$, then $\lim_{x \to -1} \frac{g_1(x)}{g_2(x)} =$ ________

5. If $h_1(x) = x^2 + 6x + 9$ and $h_2(x) = x + 3$, then $\lim_{x \to -3} \frac{h_1(x)}{h_2(x)} =$ ________

IVE. Evaluation and verification of the limit of the nth root of functions.

1. If $\lim f(x)$ exists, then $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{}$

2. If $g(x) = 5x$, then $\lim_{x \to 5} \sqrt{g(x)} =$ ________

   (a) 0 (b) 25 (c) 10 (d) 5

3. If $h(x) = x^3 + 3x^2 + 2x + 3$, then $\lim_{x \to 2} \sqrt[3]{h(x)} =$ ________

4. If $f_1(x) = x^4 - 2x^2 + 1$, then $\lim_{x \to 3} \sqrt[6]{f_1(x)} =$ ________

5. If $g_1(x) = \frac{x^2-3x-28}{x-7}$, then $\lim_{x \to 7} \sqrt[5]{g_1(x)} =$ ________

III. Evaluation and verification of limits of polynomial function.

1. $\lim_{x \to 2} (x^2 + x + 1) =$ ________

   (a) 5 (b) 7 (c) 9 (d) 11

2. $\lim_{x \to 2} x^7 =$ ______________

3. $\lim_{x \to -1} (x^5 + 3x + 4) =$ ______________

4. $\lim_{x \to 3} (4x^3 - 13x^2 + 4x + 5) =$ ______________

5. $\lim_{x \to b} \left(a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n\right) =$ ______________

II. Evaluation and verification of the limit of rational functions.

1. If $f(x) = \dfrac{x+1}{x-2}$, then $\lim_{x \to 3} f(x) =$ ______________

   (a) 0 (b) 2 (c) 4 (d) 6

2. If $h(x) = \dfrac{x^2+2x+1}{2x^2+5x}$, then $\lim_{x \to -1} h(x) =$ ______________

3. If $f(x) = \dfrac{x^2-2x+1}{x^2-1}$, then $\lim_{x \to 1} f(x) =$ ______________

4. If $g(x) = \dfrac{3x^2+13x-10}{x^2-2x-15}$, then $\lim_{x \to 5} g(x) =$ ______________

5. If $f(x) = \dfrac{x^2-x-6}{x^2+2x-15}$, for what values of a does the $\lim_{x \to a} f(x)$ not exist? ______________

I. Evaluation and verification of limits of algebraic functions.

1. If $f(x) = \dfrac{x+2}{\sqrt{x^2+4}}$, then $\lim_{x \to -2} f(x) =$ ______________

   (a) $\dfrac{\sqrt{2}}{2}$ (b) $\dfrac{2}{\sqrt{2}}$ (c) 0 (d) $\frac{1}{2}$

2. If $g(x) = \sqrt{\dfrac{x+5}{x^2+3+4}}$, then $\lim_{x \to 2} g(x) =$ ______________

3. If $h(x) = \sqrt[5]{\dfrac{x^2+7x}{x}}$, then $\lim_{x \to 0} h(x) =$ ______________

4. If $G(h) = \dfrac{\frac{1}{x} - \frac{1}{x+h}}{h}$, then $\lim\limits_{h \to 0} G(h) =$ ________________

5. If $h(x) = \dfrac{(x-1)^3}{\sqrt{x^3+3} - 2}$, then $\lim\limits_{x \to 1} h(x) =$ ________________

(hint: $\dfrac{(x-1)^3}{(\sqrt{x^2+3}-2)} = \dfrac{(x-1)^3(\sqrt{x^2+3}+2)}{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2}$ when $x \neq 1$

VITA

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Candidate for the Degree of

Doctor of Education

Thesis: A STUDY OF A LEARNING SET HIERARCHY ENCOUNTERED IN LEARNING THE CONCEPT OF THE LIMIT OF A FUNCTION

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Personal Data: Born in Odell, Nebraska, February 1, 1940, the son of Frank and Emily Hajek.

Education: Attended lower grades in rural Gage County, Nebraska; graduated from Odell High School, Odell, Nebraska in 1957; received the Bachelor of Science degree from Peru State College, Peru, Nebraska, in May, 1961; received the Master of Science degree at Oklahoma State University in May, 1966; completed requirements for the Doctor of Education degree at Oklahoma State University in May, 1970.

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